Quasi-Cross Lattice Tilings With Applications to Flash Memory

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***Abstract*—We consider lattice tilings of $\mathbb{R}^n$ by a shape we call a (k_+, k_-, n)-quasi-cross. Such lattices form perfect error-correcting codes which correct a single limited-magnitude error with prescribed maximal-magnitudes k_+ and k_- of positive error and negative error respectively (the ratio of which, $\beta = k_-/k_+$, is called the balance ratio). These codes can be used to correct both disturb and retention errors in flash memories, which are characterized by having limited magnitudes and different signs. For any rational $0 < \beta < 1$ we construct an infinite family of (k_+, k_-, n)-quasi-cross lattice tilings with balance ratio $k_-/k_+ = \beta$. We also provide a specific construction for an infinite family of $(2, 1, n)$-quasi-cross lattice tilings. The constructions are related to group splitting and modular B_1 sequences. In addition, we study bounds on the parameters of lattice-tilings by quasi-crosses, and express them in terms of the arm lengths of the quasi-crosses and the dimension. We also prove constraints on group splitting, a specific case of which shows that the parameters of the lattice tiling by $(2, 1, n)$-quasi-crosses are the only ones possible for these quasi-crosses.**

***Index Terms*—Asymmetric channel, flash memory, lattices, limited-magnitude errors, perfect codes, tiling.**

I. Introduction

FLASH memory is perhaps the fastest growing memory technology today. Flash memory cells use floating gate technology to store information using trapped charge. By measuring the charge level in a single flash memory cell and comparing it with a predetermined set of threshold levels, the charge level is quantized to one of q values, conveniently chosen to be $\mathbb{Z}_q$. While originally q was limited to be 2, and each cell stored a single bit of information, current *multilevel flash* memory technology allows much larger values of q, thus storing $\log_2 q$ bits of information in each cell. It should be noted that other alternatives have been suggested to the conventional multilevel modulation scheme, such as, for example, rank modulation [3], [13], [14], [22].

As is usually the case, the stored charge levels in flash cells suffer from noise which may affect the information retrieved from the cells. Many off-the-shelf coding solutions exist and have been applied for flash memory, see, for example, [6], [19]. However, the main problem with this approach is the fact that these codes are not tailored for the specific errors occurring in flash memory and thus are wasteful. A more accurate model of the flash memory channel is therefore required to design better-suited codes.

The most notorious property of flash memory is its inherent asymmetry between cell programming—charge injection into cells, and cell erasure—charge removal from cells (see [4]). While the former is easy to perform on single cells, the latter works on large blocks of cells and physically damages the cells. Thus, when attempting to reach a target stored value in a cell, charge is slowly injected into the cell over several iterations. If the desired level has not been reached, another round of charge injection is performed. If, however, the desired charge level has been passed, there is no way to remove the excess charge from the cell without erasing an entire block of cells. In addition, the actions of cell programming and cell reading disturb adjacent cells by injecting extra unwanted charge into them. Because the careful iterative programming procedure employs small charge-injection steps, it follows that over-programming errors, as well as cell disturbs, are likely to have a bounded magnitude of error.

This technological constraint motivated the application of the asymmetric limited-magnitude error model to the case of flash memory [5], [15]. In this model, a transmitted vector $c \in \mathbb{Z}^n$ is received with error as $y = c + e \in \mathbb{Z}^n$, where we say that t asymmetric limited-magnitude errors occurred with magnitude at most k if the error vector $e = (e_1, \dots, e_n) \in \mathbb{Z}^n$ satisfies $0 \leqslant e_i \leqslant k$ for all i, and there are exactly t nonzero entries in e. Not in the context of flash memory, it was shown in [1], and in a systematic manner in [7], how to construct optimal asymmetric limited-magnitude errors correcting *all* errors, i.e., t equals the code length. General code constructions and bounds for arbitrary t were given in [5]. More specifically, for $t = 1$, i.e., correcting a single error, codes were proposed in the context of flash in [15], but were also described in the context of semi-cross packing in the early work [10].

The main drawback of the asymmetric limited-magnitude error model is the fact that not all error types were considered during the model formulation. Another type of common error in flash memories is due to *retention* which is a slow process of charge leakage. Like before, the magnitude of errors created by retention is limited. However, unlike over-programming and cell disturbs which increase cell charge levels, retention errors reduce cell charge levels [4].

We therefore suggest a generalization to the error model we call the unbalanced limited-magnitude error model. In this model, a transmitted vector $c \in \mathbb{Z}^n$ is now received with error

Manuscript received June 23, 2011; revised October 31, 2011; accepted November 13, 2011. Date of publication November 18, 2011; date of current version March 13, 2012. The material in this paper was presented in part at the 2011 IEEE International Symposium on Information Theory. This work was supported in part by ISF Grant 134/10.

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Communicated by M. Blaum, Associate Editor for Coding Theory.
Digital Object Identifier 10.1109/TIT.2011.2176718

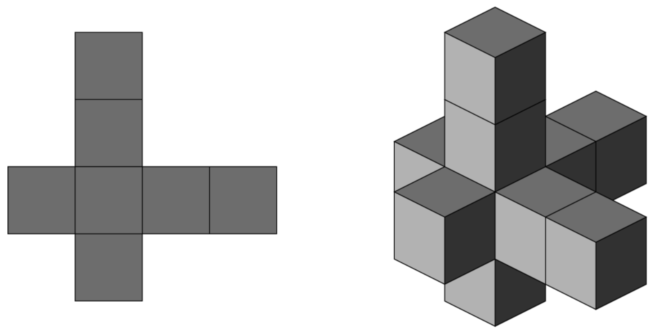

Fig. 1. A (2, 1, 2)-quasi-cross and a (2, 1, 3)-quasi-cross.

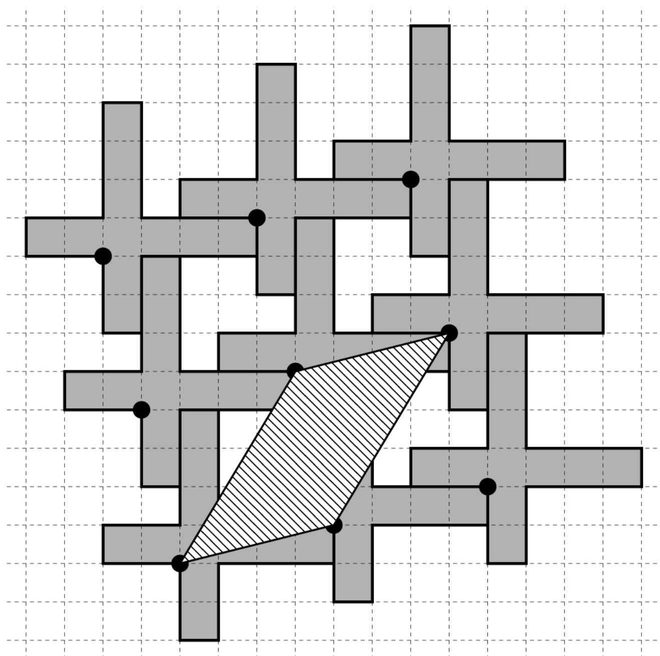

Fig. 2. Partial view of a lattice packing of a (3, 2, 2)-quasi-cross with basis $b_1 = (4,1)$, $b_2 = (3,5)$, and packing density $\frac{11}{17}$. Lattice points are marked with dots, and the hatched area is a fundamental region.

as the vector $y = c + e \in \mathbb{Z}^n$, where we say that t unbalanced limited-magnitude errors occurred if the error vector $e = (e_1, \ldots, e_n) \in \mathbb{Z}^n$ satisfies $-k_- \leqslant e_i \leqslant k_+$ for all i, and there are exactly t nonzero entries in e. Both k_+ and k_- are nonnegative integers, where we call k_+ the positive-error magnitude limit, and k_- the negative-error magnitude limit.

In this paper, we consider only single error-correcting codes. In general, assuming at most a single error occurs, the error sphere containing all possible received words $y = c + e$ forms a shape we call a (k_+, k_-, n)-*quasi-cross* (see Fig. 1). This is a generalization of the asymmetric semi-cross of [10], [15] which we get when choosing $k_- = 0$, and the full cross of [17] which we get when choosing $k_+ = k_-$. To avoid these two studied cases we shall consider only $0 < k_- < k_+$.

An error-correcting code is a packing of pair-wise disjoint quasi-crosses. We shall only consider perfect codes, i.e., tilings of the space, which form lattices, since these are easier to analyze, construct, and encode, than nonlattice packings (see Fig. 2).

The paper is organized as follows: In Section II we introduce the notation and definitions used throughout the paper and discuss connections with known results. We continue in Section III with constructions of such tilings. We follow in Section IV with simple bounds on the parameter of lattice tilings by quasi-crosses, and conclude in Section V.

II. Preliminaries

A. Quasi-Crosses, Tilings, and Lattices

In the unbalanced limited-magnitude-error channel model, the transmitted (or stored) word is a vector $v \in \mathbb{Z}^n$. A single error is a vector in $e \in \mathbb{Z}^n$ all of whose entries are 0 except for a single entry with value belonging to the set

$$M = \{-k_-, \ldots, -2, -1, 1, 2, \ldots, k_+\}$$

where the integers $0 < k_- < k_+$ are the negative-error and positive-error magnitudes. For convenience we denote this set as $M = [-k_-, k_+]^*$. We define $\beta = k_-/k_+$ and call it the *balance ratio*. Obviously, $0 < \beta < 1$.

Given a transmitted vector $v \in \mathbb{Z}^n$, and provided that at most a single error occurred, the received word resides in the error sphere centered at v and defined by

$$\mathcal{E}(v) = \{v\} \cup \{v + m \cdot e_i \mid i \in [n], m \in M\}$$

where $[n] = \{1, \ldots, n\}$, and e_i denotes the all-zero vector except for the ith position which contains a 1. We call $\mathcal{E}(0)$ a (k_+, k_-, n)-*quasi-cross*. By translation, $\mathcal{E}(v) = v + \mathcal{E}(0)$ for all $v \in \mathbb{Z}^n$.

Following the notation of [17], let

$$Q = \{(x_1, \ldots, x_n) \mid 0 \leqslant x_i < 1, x_i \in \mathbb{R}\}$$

denote the *unit cube* centered at the origin. By abuse of terminology, we shall also call the set of unit cubes $Q + \mathcal{E}(v)$, a (k_+, k_-, n)-quasi-cross centered at v for any $v \in \mathbb{Z}^n$. Examples of such quasi-crosses are given in Fig. 1. We note that the volume of $Q + \mathcal{E}(v)$ does not depend on the choice of v and is equal to $n(k_+ + k_-) + 1$.

A set $V = \{v_1, v_2, \ldots\} \subseteq \mathbb{Z}^n$ defines a set of quasi-crosses by translation: $\{\mathcal{E}(v_1), \mathcal{E}(v_2), \ldots\}$. The set V is said to be a *packing* of $\mathbb{R}^n$ by quasi-crosses if the translated quasi-crosses are pairwise disjoint. A packing V is called a *tiling* if the union of the translated quasi-crosses equals $\mathbb{R}^n$. If V is an additive subgroup of $\mathbb{Z}^n$ with a basis $\{b_1, b_2, \ldots, b_n\}$, then we call V a *lattice*. The $n \times n$ integer matrix formed by placing the elements of a basis as its rows is called a *generating matrix* of the lattice.

Let $\Lambda \subseteq \mathbb{Z}^n$ be a lattice with a generating matrix $\mathcal{G}(\Lambda) \in \mathbb{Z}^{n \times n}$ whose rows form a basis $\{b_1, b_2, \ldots, b_n\} \subseteq \mathbb{Z}^n$. A fundamental region of Λ is defined as

$$\left\{ \sum_{i=1}^{n} \alpha_i b_i \;\middle|\; \alpha_i \in \mathbb{R}, 0 \leqslant \alpha_i < 1 \right\}.$$

It is easily seen, by definition, that Λ tiles $\mathbb{R}^n$ with translates of the fundamental region.

It is well known that the volume of a fundamental region does not depend on the choice of basis for Λ and equals $\det \mathcal{G}(\Lambda)$. The *density* of Λ is defined as $1/\det \mathcal{G}(\Lambda)$ and if Λ forms a packing of (k_+, k_-, n)-quasi-crosses, then the *packing density* of Λ is defined as

$$\rho(\Lambda) = \frac{n(k_+ + k_-) + 1}{\det \mathcal{G}(\Lambda)}$$

which intuitively measures (for a large enough finite area) the ratio of the area covered by (k_+, k_-, n)-quasi-crosses centered at the lattice points, to the total area. It follows that $0 \leqslant \rho(\Lambda) \leqslant 1$, and Λ forms a tiling with (k_+, k_-, n)-quasi-crosses if and only if $\rho(\Lambda) = 1$, i.e., $\det \mathcal{G}(\Lambda) = n(k_+ + k_-) + 1$.

Example 1: If we take the (3, 2, 2)-quasi-cross, one can verify that the lattice Λ with a generating matrix

$$G(\Lambda) = \begin{pmatrix} 4 & 1 \\ 3 & 5 \end{pmatrix}$$

is indeed a lattice packing for this quasi-cross (see Fig. 2). The resulting packing density is

$$\rho(\Lambda) = \frac{2(3+2)+1}{\det \mathcal{G}(\Lambda)} = \frac{11}{17}.$$

□

As a final note, it is interesting to point out that the $(1, 1, n)$-quasi-cross (which is in fact a full cross), is also a Lee sphere of radius 1. The question of whether Lee spheres can be tiled dates back to the work of Golomb and Welch [9], and can be found in other more recent works [8], [11], [12], [16].

B. Application to Flash Memory

At this point we stop to ponder our choice for using $\mathbb{Z}^n$ as the space of transmitted (or stored) messages. This choice certainly makes the analysis of tilings simpler. However, a single multi-level flash cell is only capable of storing a single value from $\mathbb{Z}_q$. Thus, from a lattice $\Lambda \subseteq \mathbb{Z}^n$ we may construct a code

$$\mathcal{C} = \Lambda \cap \mathbb{Z}_q^n,$$

whose codewords can be stored in n multilevel flash cells with q levels each.

We note that even if Λ is a tiling by (k_+, k_-, n)-quasi-crosses, then $\mathcal{C}$ is not necessarily a perfect code (in the sense that the error-spheres around the codewords form a partition of the space $\mathbb{Z}_q^n$). That is due to the fact that the arms of some quasi-crosses centered around points outside $\mathbb{Z}_q^n$, may extend into $\mathbb{Z}_q^n$. However, it is still possible to get a good bound from below on the rate of the code in the following way: For any vector $v \in \mathbb{Z}^n$, the coset $v + \Lambda$ is also a tiling by (k_+, k_-, n)-quasi-crosses (simply by virtue of being a geometric translate of Λ). We now define

$$\mathcal{C}_v = (v + \Lambda) \cap \mathbb{Z}_q^n$$

and a standard averaging argument guarantees the existence of some $v \in \mathbb{Z}^n$ for which

$$|\mathcal{C}_v| = \left|(v + \Lambda) \cap \mathbb{Z}_q^n\right| \geqslant \frac{q^n}{n(k_+ + k_-) + 1}.$$

The best rate of some coset of the code is then bounded from below by

$$R(\mathcal{C}_v) = \frac{\log |\mathcal{C}_v|}{\log q^n} \geqslant 1 - \frac{1}{n} \log_q \left(n(k_+ + k_-) + 1\right). \quad (1)$$

We say a lattice $\Lambda \subseteq \mathbb{Z}^n$ has *period* $(t_1, \dots, t_n) \in \mathbb{Z}^n$ if whenever $v \in \Lambda$, then also $v + t_i e_i \in \Lambda$ for all i. Lattices are always periodic, and the *minimal period* in the ith direction, t_i, is the smallest positive integer for which $t_i e_i \in \Lambda$. By taking $t = \mathrm{lcm}(t_1, t_2, \dots, t_n)$, we can say the lattice has period t in all directions.

Finally, assume Λ is a packing of $\mathbb{R}^n$ by (k_+, k_-, n)-quasi-crosses. Then the code $\mathcal{C}$ defined above is capable of correcting a single unbalanced limited-magnitude error. If, in addition, the number of levels, q, is a multiple of a period t of Λ, then the code $\mathcal{C}$ is capable of correcting a single error with wrap-around, i.e., the error vector is added to the codeword over $\mathbb{Z}_q^n$ (and not over $\mathbb{Z}^n$).

C. Lattice Tiling via Group Splitting

We continue with our treatment of lattice tilings. An equivalence between lattice packings and group splitting was described in [10], [17], which we describe here for completeness. Let G be an Abelian group, where we shall denote the group operation as $+$. Given some $s \in G$ and a nonnegative integer $m \in \mathbb{Z}$, we denote by ms the sum $s + s + \cdots + s$, where s appears in the sum m times. The definition is extended in the natural way to negative integers m.

A splitting of G is a pair of sets, $M \subseteq \mathbb{Z} \setminus \{0\}$, called the *multiplier set*, and $S = \{s_1, s_2, \dots, s_n\} \subseteq G$, called the *splitter set*, such that the elements of the form ms, $m \in M$, $s \in S$, are all distinct and nonzero in G. Next, we define a homomorphism $\phi : \mathbb{Z}^n \to G$ by

$$\phi(x_1, x_2, \dots, x_n) = \sum_{i=1}^{n} x_i s_i.$$

If the multiplier set is $M = [-k_-, k_+]^*$, then it may be easily verifiable that

$$\ker \phi = \{(x_1, x_2, \dots, x_n) \in \mathbb{Z}^n \mid \phi(x_1, x_2, \dots, x_n) = 0\}$$

is a lattice packing of $\mathbb{R}^n$ by (k_+, k_-, n)-quasi-crosses. That $\ker \phi$ is a lattice is obvious. To show that the lattice is a packing of (k_+, k_-, n)-quasi-crosses, assume to the contrary two such distinct quasi-crosses, one centered at $x = (x_1, \dots, x_n)$ and one centered at $y = (y_1, \dots, y_n)$, have a nonempty intersection, i.e., $x + m_1 e_i = y + m_2 e_j$, where $m_1, m_2 \in M$, then

$$m_1 s_i = \phi(x + m_1 e_i) = \phi(y + m_2 e_j) = m_2 s_j$$

which is possible only if $m_1 = m_2$ and $i = j$, resulting in the two quasi-crosses being the same one—a contradiction. The packing is a tiling iff $|G| = n(k_+ + k_-) + 1$.

A simple representation of the lattice may also be given in a matrix form: Let $\mathcal{H} = [s_1, s_2, \dots, s_n]$ be a $1 \times n$ matrix over G. The lattice Λ is the set of vectors $x = (x_1, \dots, x_n) \in \mathbb{Z}^n$

such that $\mathcal{H}x^T = 0$. Thus, $\mathcal{H}$ plays the role of a "parity-check matrix."

Example 2: Continuing Example 1, let $G = \mathbb{Z}_{17}$ and let $M = \{-2, -1, 1, 2, 3\} = [-2, 3]^*$ stand for the multiplier set of the $(3, 2, n)$-quasi-cross. A possible splitting of G is $S = \{1, 13\}$, which results in a parity-check matrix $\mathcal{H} = [1, 13]$ for the packing described in Example 1. □

Group splitting as a method for constructing error-correcting codes was also discussed, for example, in the case of shift-correcting codes [20] and integer codes [21].

D. Lattice Packings and Sequences

It was noted in [15] that there is a connection between the codes constructed in [15] (which are equivalent to semi-cross packings) and a certain subcase of sequences called modular B_h sequences. We detail the nature of the connection relevant to our case.

A v-modular $B_h(M)$ sequence, where $M \subseteq \mathbb{Z} \setminus \{0\}$, is a subset[1] $S \subseteq \mathbb{Z}_v \setminus \{0\}$, whose elements $S = \{s_1, \dots, s_n\}$ satisfy that all the sums $\sum_{i=1}^{h} m_i s_{j_i}$ are distinct, where $1 \leqslant j_1 < j_2 < \dots < j_h \leqslant n$, and $m_i \in M$.

Thus, a v-modular $B_1(M)$ sequence is a splitting of $\mathbb{Z}_v$ defined by M and S. We stress that these sequences are defined only by splitting a cyclic group.

As was also described in [15], when we have a v-modular $B_1(M)$ sequence S, i.e., a splitting of $\mathbb{Z}_v$ by M and S, and therefore, a resulting $1 \times n$ parity-check matrix $\mathcal{H} = [s_1, s_2, \dots, s_n]$, we can construct other packings, provided the elements of M are co-prime to v. This is done by constructing any $k \times n(v^k - 1)/(v - 1)$ parity-check matrix $\mathcal{H}'$ containing all distinct column vectors whose top nonzero element is from S. This is equivalent to a splitting of the noncyclic group $\mathbb{Z}_v^k$ by M and S being the columns of $\mathcal{H}'$. We note that if $\mathcal{H}$ results in a tiling, then so does $\mathcal{H}'$.

III. Constructions of Tilings by Quasi-Crosses

We shall now consider constructions of lattice tilings by (k_+, k_-, n)-quasi-crosses. We first examine the case of a constant balance ratio $0 < \beta < 1$ and show that for any rational β there exist infinitely many triplets (k_+, k_-, n) such that $k_-/k_+ = \beta$ and the (k_+, k_-, n)-quasi-crosses tile. This is accomplished by constructions for all $k_+ + k_- = p - 1$, where p is a prime. We then focus on a particular case of $(2, 1, n)$-quasi-crosses and show an infinite family of tilings for them. We shall conclude this section by describing simple encoding and decoding algorithms.

A. Constant Balance-Ratio Quasi-Cross Tilings

Construction 1: Let $0 < k_- < k_+$ be positive integers such that $k_+ + k_- = p - 1$, where p is a prime. We set the multiplier set $M = [-k_-, k_+]^*$. Consider the cyclic group $G = \mathbb{Z}_{p^\ell}$, $\ell \in \mathbb{N}$. We split G using a splitter set S constructed recursively in the following manner:

$$S_1 = \{1\}$$
$$S_{i+1} = pS_i \cup \left\{ s \in \mathbb{Z}_{p^{i+1}} \;\middle|\; s \equiv 1 \pmod{p} \right\}.$$

The requested set is $S = S_\ell$.

Theorem 3: The sets S and M from Construction 1 split $\mathbb{Z}_{p^\ell}$, forming a tiling by $(k_+, k_-, (p^\ell - 1)/(p-1))$-quasi-crosses and a p^ℓ-modular $B_1(M)$ sequence.

Proof: The proof is by a simple induction. Obviously M and $S_1 = \{1\}$ split $\mathbb{Z}_p$. Now assume M and S_i split $\mathbb{Z}_{p^i}$. Let us consider M, S_{i+1}, and $\mathbb{Z}_{p^{i+1}}$. We now show that if $ms = m's'$ in $\mathbb{Z}_{p^{i+1}}$, m, $m' \in M$, s, $s' \in S_{i+1}$, then $m = m'$ and $s = s'$.

In the first case, given any $s \in S_{i+1}$, $p \nmid s$, and given m, $m' \in M$, $m \neq m'$, since $M = [-k_-, k_+]^*$, it follows that $ms \neq m's$ since they leave different residues modulo p.

For the second case, let s, $s' \in S$, $s' \neq s$, $p \nmid s$, and let m, $m' \in M$, where m and m' are not necessarily distinct. If $p | s'$ then $ms \neq m's'$ since $p \nmid ms$ but $p | m's'$. We assume then that $s' \equiv 1 \pmod{p}$. Write $s = qp + 1$ and $s' = q'p + 1$, $0 \leqslant q$, $q' \leqslant p^i - 1$, then $ms = m's'$ implies $m = m'$ (by reduction modulo p). It then follows that $mqp \equiv mq'p \pmod{p^{i+1}}$. But $\gcd(m, p) = 1$ and so $q \equiv q' \pmod{p^i}$, which (due to the range of q and q') implies $q = q'$, i.e., $s = s'$.

For the last case, s, $s' \in pS_i$. We note that the multiples of p in $\mathbb{Z}_{p^{i+1}}$ are isomorphic to $\mathbb{Z}_{p^i}$, and since M and S_i split $\mathbb{Z}_{p^i}$, for all m, $m' \in M$, if $ms = m's'$ then $m = m'$ and $s = s'$.

Finally, $|M| = p - 1$, $|S_\ell| = (p^\ell - 1)/(p - 1)$, and so $|M| \cdot |S_\ell| + 1 = \left|\mathbb{Z}_{p^\ell}\right|$, implying that the splitting induces a tiling. ■

The following construction splits a noncyclic group of the same parameters.

Construction 2: Let $0 < k_- < k_+$ be positive integers such that $k_+ + k_- = p - 1$, where p is a prime. We set the multiplier set $M = [-k_-, k_+]^*$. Consider the additive group of $G = \mathrm{GF}(p^\ell)$, $\ell \in \mathbb{N}$. Let $\alpha \in \mathrm{GF}(p^\ell)$ be a primitive element, and define $S = \{P(\alpha) \mid P \in \mathcal{M}_\ell^p[x]\}$ where $\mathcal{M}_\ell^p[x]$ denotes the set of all monic polynomials of degree strictly less than $\ell - 1$ over $\mathrm{GF}(p)$ in the indeterminate x.

Theorem 4: The sets S and M from Construction 2 split the additive group of $\mathrm{GF}(p^\ell)$ and form a tiling by $(k_+, k_-, (p^\ell - 1)/(p - 1))$-quasi-crosses.

Proof: Since α is primitive in $\mathrm{GF}(p^\ell)$, the elements $1, \alpha, \alpha^2, \dots, \alpha^{\ell-1}$ form a basis of the additive group of $\mathrm{GF}(p^\ell)$ over $\mathrm{GF}(p)$. Since $M = \mathrm{GF}^*(p)$, it is easily seen that $ms = m's'$, m, $m' \in M$, s, $s' \in S$, implies $m = m'$ and $s = s'$. Again, by counting the size of M and S, the splitting induces a tiling. ■

We point out several interesting observations. In Construction 2, if we take $\ell = 1$ we get $S = \{1\}$. For $\ell > 1$, write the elements of $\mathrm{GF}(p^\ell)$ as length-ℓ vectors over $\mathrm{GF}(p)$ (using the basis $1, \alpha, \dots, \alpha^{\ell-1}$, with α a primitive element of $\mathrm{GF}(p^\ell)$). The elements of S then become the set of all vectors of length ℓ over $\mathrm{GF}(p)$ with the leading nonzero element being 1. We will get the same set by extending the "matrix-extension" method implied in [15] to our quasi-cross case.

[1]The actual sequence is the binary characteristic sequence of the subset to be defined shortly.

Another interesting thing to note is that, using the same vector notation as above, the parity-check matrix for the lattice is simply the parity-check matrix of the $[\frac{p^\ell-1}{p-1}, \frac{p^\ell-1}{p-1}-\ell, 3]$ Hamming code over $\mathrm{GF}(p)$.

Yet another observation is that we can mix Constructions 1 and 2, by taking the p^ℓ-modular $B_1(M)$ sequence resulting from Construction 1 and applying the "matrix" method of Construction 2 to form a splitting of $G = \mathbb{Z}_{p^\ell} \times \mathbb{Z}_{p^\ell} \times \ldots \times \mathbb{Z}_{p^\ell}$ which induces a tiling by quasi-crosses. The latter works since the elements of M are all co-prime to p.

Finally, as is shown in the next example, we observe that the lattice tilings resulting from Constructions 1 and 2 are not equivalent.

Example 5: Consider six-dimensional lattice tilings by $(3,1,6)$-quasi-crosses. Using Construction 1 we construct a lattice Λ_1 by splitting $\mathbb{Z}_{25}$ and getting a splitter set $S = \{1,5,6,11,16,21\}$, resulting in a parity-check matrix

$$\mathcal{H}_1 = [1 \quad 5 \quad 6 \quad 11 \quad 16 \quad 21]$$

over $\mathbb{Z}_{25}$. This produces a generating matrix for Λ_1

$$\mathcal{G}_1 = \begin{bmatrix} 25 & 0 & 0 & 0 & 0 & 0 \\ 20 & 1 & 0 & 0 & 0 & 0 \\ 19 & 0 & 1 & 0 & 0 & 0 \\ 14 & 0 & 0 & 1 & 0 & 0 \\ 9 & 0 & 0 & 0 & 1 & 0 \\ 4 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

We confirm that

$$\det \mathcal{G}_1 = 25 = 6(3+1)+1$$

making Λ_1 a tiling for (3, 1, 6)-quasi-crosses.

If, on the other hand, we choose to use Construction 2 to construct a lattice Λ_2, we split $\mathrm{GF}(5^2)$ to get a parity-check matrix

$$\mathcal{H}_2 = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 2 & 3 & 4 \end{bmatrix}$$

over $\mathrm{GF}(5)$. A corresponding generating matrix is then

$$\mathcal{G}_2 = \begin{bmatrix} 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 & 0 \\ 4 & 4 & 1 & 0 & 0 & 0 \\ 3 & 4 & 0 & 1 & 0 & 0 \\ 2 & 4 & 0 & 0 & 1 & 0 \\ 1 & 4 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Again, we confirm $\det \mathcal{G}_2 = 25$. On a side note, the code $\mathcal{C}_2 = \Lambda_2 \cap \mathbb{Z}_5^6$ is none other than the $[6,4,3]$ Hamming code over $\mathrm{GF}(5)$.

Finally, to show the lattices are not equivalent, it is readily verified that the minimal period of Λ_1 is (25, 5, 25, 25, 25, 25), while the minimal period of Λ_2 is (5, 5, 5, 5, 5, 5). ■

The following shows there are infinitely many tilings by quasi-crosses of any given rational balance ratio.

Theorem 6: For any given rational balance ratio $\beta = k_-/k_+$, $0 < \beta < 1$, there exists an infinite sequence of quasi-crosses, $\{(k_+^{(i)}, k_-^{(i)}, n^{(i)})\}_{i=1}^{\infty}$, such that $n^{(i)} < n^{(i+1)}$, $k_-^{(i)}/k_+^{(i)} = \beta$, and there exists a tiling by $(k_+^{(i)}, k_-^{(i)}, n^{(i)})$-quasi-crosses, for all $i \in \mathbb{N}$.

Proof: Given a rational $0 < \beta < 1$, let $k_+, k_- \in \mathbb{N}$ be such that $k_-/k_+ = \beta$. Denote $d = k_+ + k_-$ and consider the arithmetic progression $1, 1+d, 1+2d, \ldots, 1+id, \ldots$. Since $\gcd(1,d) = 1$, by Dirichlet's Theorem (see, for example, [2]), the sequence contains infinitely many prime numbers. For any such prime, p, there exists $q \in \mathbb{N}$ such that $qk_+ + qk_- = p-1$. We can then apply Constructions 1 and 2 to form tilings by (qk_+, qk_-, n)-quasi-crosses with the required balance ratio and n unbounded. ■

B. Construction of $(2,1,n)$-Quasi-Cross Tilings

We turn to constructing $(2,1,n)$-quasi-cross tilings and their associated modular $B_1(M)$ sequences. The construction is similar in flavor to Construction 1.

Construction 3: Let $k_+ = 2$, $k_- = 1$, and let the multiplier set be $M = \{-1, 1, 2\}$. We split the group $G = \mathbb{Z}_{4^\ell}$, $\ell \in \mathbb{N}$, using a splitter set S constructed recursively in the following manner:

$$S_1 = \{1\}$$
$$S_{i+1} = 4S_i \cup \left\{ s \in \mathbb{Z}_{4^{i+1}} \mid s \equiv 1 \pmod{2}, 2s < 4^{i+1} \right\}$$

The requested set is $S = S_\ell$.

Theorem 7: The sets S and M from Construction 3 split $\mathbb{Z}_{4^\ell}$, forming a tiling by $(2,1,(4^\ell-1)/3)$-quasi-crosses and a 4^ℓ-modular $B_1(M)$ sequence.

Proof: The proof is by induction. The sets M and S_1 obviously split $\mathbb{Z}_4$. Assume M and S_i split $\mathbb{Z}_{4^i}$ and consider M and S_{i+1}. For convenience, denote

$$S'_{i+1} = \left\{ s \in \mathbb{Z}_{4^{i+1}} \mid s \equiv 1 \pmod{2}, 2s < 4^{i+1} \right\}.$$

It is easily seen that due to the restriction $2s < 4^{i+1}$, the elements of S'_{i+1} and $-S'_{i+1}$ are distinct, and together they contain all the odd integers in $\mathbb{Z}_{4^{i+1}}$. The elements of $2S'_{i+1}$ are then also distinct and contain all the even integers in $\mathbb{Z}_{4^{i+1}}$ leaving a residue of 2 modulo 4.

We are then left with all the multiples of 4 in $\mathbb{Z}_{4^{i+1}}$ which form a group isomorphic to $\mathbb{Z}_{4^i}$, and thus, by the induction hypothesis, are split by M and $4S_i$.

A simple counting argument shows that $|M| = 3$, $|S_\ell| = \frac{4^\ell-1}{3}$, and therefore $|M|\,|S_\ell| + 1 = |\mathbb{Z}_{4^\ell}|$. It follows that M and S_ℓ split $\mathbb{Z}_{4^\ell}$ and form a tiling. ■

We observe that in this case, since the elements of M are not co-prime to 4, extending the matrix method from [15] does not produce a valid tiling or even packing. For example, if we were to take the trivial 4-modular $B_1(M)$ sequence, $\{1\}$ and attempt to create a parity-check matrix over $\mathbb{Z}_4$

$$\mathcal{H} = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 2 & 3 \end{bmatrix}$$

we would find that M together with the columns of $\mathcal{H}$ is not a splitting of $\mathbb{Z}_4^2$ since $2 \cdot [1,0]^T = 2 \cdot [1,2]^T$ over $\mathbb{Z}_4$. Hence, the lattice formed by the parity-check matrix $\mathcal{H}$ is not a lattice packing of (2, 1, 5)-quasi-crosses.

Example 8: To find a tiling by (2, 1, 5)-quasi-crosses using Construction 3, we construct a lattice Λ by splitting $\mathbb{Z}_{16}$ with $S = \{1, 3, 4, 5, 7\}$. The parity-check and generating matrices are

$$\mathcal{H} = [1 \quad 3 \quad 4 \quad 5 \quad 7], \qquad \mathcal{G} = \begin{bmatrix} 16 & 0 & 0 & 0 & 0 \\ 13 & 1 & 0 & 0 & 0 \\ 12 & 0 & 1 & 0 & 0 \\ 11 & 0 & 0 & 1 & 0 \\ 9 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Unlike Λ_2 from Example 5, which turned out to be a Hamming code in disguise, the lattice Λ does not seem to be related to Hamming codes, or 1-error-correcting codes in the Hamming metric: Its minimal period is (16, 16, 4, 16, 16), and it contains a lattice point (2, 0, 0, 0, 2) of Hamming weight 2. □

C. Encoding and Decoding Algorithms

We first consider decoding of lattice-based codes. Let Λ be a tiling by (k_+, k_-, n)-quasi-crosses constructed from a splitting of the Abelian group G by $M = [-k_-, k_+]^*$ and $S = \{s_1, s_2, \dots, s_n\}$. Let $\mathcal{H} = [s_1, s_2, \dots, s_n]$ be a parity-check matrix for Λ. Assume some $c \in \Lambda$ was stored as a codeword, but $y = c + e$ was received, where e is a single-error vector, $e = m \cdot e_i$ for some $m \in M$. Thus, y resides in the (k_+, k_-, n)-quasi-cross centered around c. We define the *syndrome* of y as

$$\sigma(y) = Hy^T = H(c+e)^T = He^T = ms_i.$$

Obviously, if $\sigma(y) = 0$ then $c = y$ and no correction is required. In all other cases, while we can calculate $\sigma(y) \in G$, we may not be immediately aware of its unique representation as ms_i (guaranteed by the splitting). However, once we are able to find m and s_i, then correcting the received word is easily done by noting that $c = y - me_i$.

In a practical setting we usually have k_+, k_-, and q, which are all constant (depending on the technological constraints of the implementing hardware), while the code length n may vary and be much larger. In that case, the number of possible syndromes (also the volume of a (k_+, k_-, n)-quasi-cross) is $O(n)$. Thus, exhaustively checking all possible pairs m and s_i to find the unique pair for which $\sigma(y) = ms_i$, takes $O(n)$ time. We cannot do any better (in terms of complexity) since just calculating the syndrome itself takes $O(n)$ operations.

Despite the fact that we cannot improve upon the decoding complexity, we briefly present simple algorithms for decomposing $\sigma(y)$ into ms_i. The algorithms differ, depending on the construction used. We shall use the following notation: for any $a, b \in \mathbb{N}$ let $\xi_b(a)$ denote the largest integer such that $\xi_b(a) | a$ but $b \nmid \xi_b(a)$. We shall also use the notation of Constructions 1–3.

- **Construction 1:** We set
$$m = \xi_p(\sigma(y)) \bmod p, \qquad s_i = \frac{\sigma(y)}{m}$$
where division by m is done over $\mathbb{Z}_{p^\ell}$.
- **Construction 2:** If $\sigma(y) \in \mathrm{GF}(p^\ell)$ is written as a length-ℓ vector over $\mathrm{GF}(p)$ using the powers of α as a basis, then m is the vector element corresponding to the highest degree of α with nonzero coefficient. After finding m, we also find $s_i = \sigma(y)/m$, where the division is over $\mathrm{GF}(p^\ell)$.
- **Construction 3:** We set
$$m = \begin{cases} 1 & \xi_4(\sigma(y)) \text{ is odd, and } \sigma(y) < 4^\ell/2 \\ -1 & \xi_4(\sigma(y)) \text{ is odd, and } \sigma(y) \geqslant 4^\ell/2 \\ 2 & \xi_4(\sigma(y)) \text{ is even} \end{cases}$$
$$s_i = \begin{cases} \sigma(y) & m = 1 \\ 4^\ell - \sigma(y) & m = -1 \\ \sigma(y)/2 & m = 2 \end{cases}$$
where division by 2 is over $\mathbb{Z}$.

Unlike the simplicity of the decoding procedure, encoding is more complicated. This is mainly due to the fact that finding the exact intersection or even its size [see the discussion leading to (1)] is not easy. A brute-force approach can be employed, in which we list all the elements in $\Lambda \cap \mathbb{Z}_q^n$ and form a codebook mapping input vectors to codewords. This, however, is impractical for large values of n since the codebook's size grows exponentially with n.

We now suggest a simple systematic encoding procedure which is applicable to *subcodes* of Constructions 1 and 3. The procedure's simplicity, however, comes at a price of a lower code rate. Let the construction split $\mathbb{Z}_{a^\ell}$ (a being either a prime in Construction 1, or 4 in Construction 3), and we further require that $q \geqslant a^\ell$. The splitting set $S = \{s_1, \dots, s_n\}$ in both constructions always contains the value 1, and w.l.o.g., we assume $s_n = 1$. The encoding function

$$E : \mathbb{Z}_q^{n-1} \to \Lambda \cap \mathbb{Z}_q^n$$

is defined as

$$E(u_1, u_2, \dots, u_{n-1}) = (u_1, u_2, \dots, u_{n-1}, b)$$

where

$$b = \left(a^\ell - \sum_{i=1}^{n-1} s_i u_i \right) \bmod a^\ell.$$

One can easily verify the image of the encoding function E is contained in $\Lambda \cap \mathbb{Z}_q^n$, and is therefore a subcode of Construction 1 or 3. Since the encoding is systematic, the reverse mapping is simple. Finally, the rate of the proposed subcodes is $1 - \frac{1}{n}$, which coincides with (1) when $q = n(k_+ + k_-) + 1$, but is otherwise lower.

IV. Bounds on the Parameters of Lattice Tilings by Quasi-Crosses

In this section we focus on showing bounds on the parameters of (k_+, k_-, n)-quasi-cross tilings. We first consider the restrictions (k_+, k_-, n)-quasi-cross tilings imply on k_+, k_-, and n. We then continue to study the group G being split to create the tilings, and show restrictions which, in particular, prove that the parameters of the $(2, 1, n)$-quasi-cross tiling of Construction 3 are unique.

A. Dimension and Arm Length Bounds

We first discuss bounds on the parameters of lattice-tilings by quasi-crosses, expressed in terms of the arm lengths of the quasi-crosses and the dimension of the tiling. Some of the theorems to follow may be viewed as extensions to [18].

Theorem 9: For any $n \geqslant 2$, if

$$\frac{2k_+(k_-+1)-k_-^2}{k_++k_-} > n$$

then there is no lattice tiling of (k_+, k_-, n)-quasi-crosses.

Proof: Given an integer $n \geqslant 2$, assume a (k_+, k_-, n)-quasi-cross lattice tiling Λ exists. Consider the plane $\{(x, y, 0, \dots, 0) \mid x, y \in \mathbb{Z}\}$. Translates of this plane tile $\mathbb{Z}^n$. Within this plane, we look at the subset

$$A = \{(x, y, 0, \dots, 0) \mid 0 \leqslant x, y < k_+ + 2 \text{ and } x < k_- + 2 \text{ or } y < k_- + 2\}.$$

It is easily seen that A cannot contain two points from Λ, or else the arms of two quasi-crosses overlap. Thus, the density of Λ (which we know is exactly $1/(n(k_+ + k_-) + 1)$, since Λ is a tiling) cannot exceed the reciprocal of the volume of A, i.e.,

$$\frac{1}{n(k_++k_-)+1} \leqslant \frac{1}{(k_++1)^2-(k_+-k_-)^2}.$$

Rearranging gives us the desired result. ∎

Corollary 10: There is no lattice tiling of $\mathbb{R}^2$ by $(k_+, k_-, 2)$-quasi-crosses.

Proof: It is easily verifiable that for any $0 < k_- < k_+$,

$$\frac{2k_+(k_-+1)-k_-^2}{k_++k_-} > 2.$$

∎

In the following theorem and corollary we can restrict the arm lengths of quasi-crosses that lattice-tile $\mathbb{R}^n$.

Theorem 11: For any $n \geqslant 2$, if a lattice tiling of $\mathbb{R}^n$ by (k_+, k_-, n)-quasi-crosses exists, then $k_- \leqslant n - 1$.

Proof: Let $0 < k_- < k_+$, and let $M = [-k_-, k_+]^*$. Assume there is a splitting of an Abelian group G by M and $S = \{s_1, \dots, s_n\}$ which induces a lattice tiling by (k_+, k_-, n)-quasi-crosses, i.e., $|G| = n(k_+ + k_-) + 1$.

We first claim that for all $2 \leqslant i \leqslant n$ there are integers x_i and y_i such that

$$k_+ + 1 \leqslant x_i \leqslant \left\lfloor \tfrac{n(k_++k_-)+1}{k_-+1} \right\rfloor$$
$$|y_i| \leqslant k_-$$
$$s_1 x_i + s_i y_i = 0.$$

To prove this, fix i and let us look at the integers

$$0 \leqslant a_1 \leqslant \left\lfloor \frac{n(k_++k_-)+1}{k_-+1} \right\rfloor, \qquad 0 \leqslant a_2 \leqslant k_-$$

and the sums $s_1 a_1 + s_i a_2$. Since

$$\left(\left\lfloor \frac{n(k_++k_-)+1}{k_-+1} \right\rfloor + 1\right)(k_-+1) \geqslant$$
$$\geqslant n(k_++k_-)+1-k_-+k_-+1$$
$$= n(k_++k_-)+2 > |G|$$

by the pigeonhole principle there exist two distinct pairs, b_1, b_2, and c_1, c_2, such that

$$s_1 b_1 + s_i b_2 = 0 \qquad s_1 c_1 + s_i c_2 = 0.$$

Assume w.l.o.g. that $b_1 \geqslant c_1$ and define

$$d_1 = b_1 - c_1 \qquad d_2 = b_2 - c_2.$$

We now get $s_1 d_1 + s_i d_2 = 0$, where $(d_1, d_2) \neq (0, 0)$. In addition

$$0 \leqslant d_1 \leqslant \left\lfloor \frac{n(k_++k_-)+1}{k_-+1} \right\rfloor, \qquad |d_2| \leqslant k_-.$$

If $0 \leqslant d_1 \leqslant k_+$ then $s_1 d_1 = -s_i d_2$ contradicts the fact that S and M split G. Thus,

$$k_+ + 1 \leqslant d_1 \leqslant \left\lfloor \frac{n(k_++k_-)+1}{k_-+1} \right\rfloor,$$

which proves our claim regarding the existence of x_i and y_i.

For the rest of the proof we distinguish between two cases.

Case 1: There exist $i \neq j$ such that $x_i = x_j$. In that case

$$0 = s_1 x_i + s_i y_i = s_1 x_j + s_j y_j$$

in which case, $0 = s_i y_i = s_j y_j$. However, $-k_- \leqslant y_i, y_j \leqslant k_-$ and to avoid contradicting the splitting, necessarily $y_i = y_j = 0$. It follows that $s_1 x_i = 0$. We now note that

$$-k_- s_1, \dots, -s_1, 0, s_1, \dots, k_+ s_1$$

are all distinct, and so the order of s_1 in G is at least $k_+ + k_- + 1$, but has to divide x_i. Hence,

$$k_+ + k_- + 1 \leqslant x_i \leqslant \left\lfloor \frac{n(k_++k_-)+1}{k_-+1} \right\rfloor.$$

Rearranging the two sides gives us

$$k_- \leqslant n - 1 - \frac{k_-}{k_++k_-}$$

and since $0 < k_- < k_+$, necessarily $k_- \leqslant n - 2$.

Case 2: If $i \neq j$, then $x_i \neq x_j$. Thus, the number of distinct values does not exceed their range, and we get

$$n - 1 \leqslant \left\lfloor \frac{n(k_++k_-)+1}{k_-+1} \right\rfloor - k_+.$$

Rearranging this we get

$$k_- \leqslant n - 1 + \frac{1}{k_+ - 1}.$$

If $k_+ > 2$ then, by the above, $k_- \leqslant n - 1$. If, however, $k_+ = 2$, then $k_- = 1$ and obviously $k_- \leqslant n - 1$. ∎

Corollary 12: For any $n \geqslant 3$, if a lattice tiling of $\mathbb{R}^n$ by (k_+, k_-, n)-quasi-crosses exists and $k_- > \frac{n}{2} - 1$, then

$$k_+ \leqslant \begin{cases} \frac{3n^2}{8} & n \text{ is even,} \\ \frac{3n^2-4n+1}{4} & n \text{ is odd.} \end{cases}$$

Proof: By Theorem 9, a necessary condition for a lattice tiling to exist is that

$$\frac{2k_+(k_-+1)-k_-^2}{k_++k_-} \leqslant n$$

or after rearranging

$$k_+(2(k_- + 1) - n) \leqslant k_-^2 + nk_-.$$

If $k_- > \frac{n}{2} - 1$, the left-hand side (LHS) is positive and we get

$$k_+ \leqslant \frac{k_-^2 + nk_-}{2(k_- + 1) - n}.$$

We need to maximize k_+, and by Theorem 11 we can restrict ourselves to $k_- \leqslant n-1$. The maximum is achieved at $k_- = \frac{n}{2}$ for n even, and at $k_- = \frac{n-1}{2}$ for n odd. Substituting back into the bound on k_+ gives the desired result. ■

B. Restrictions on the Split Group

We now turn to examining connections between properties of the Abelian group being split, G, and the multiplier and splitter sets, M and S. We shall eventually show, as a special case of the theorems presented, that the $(2,1,n)$-quasi-cross tiles $\mathbb{R}^n$ only with the parameters of Construction 3. We follow the notation and definitions of [18].

Definition 13: Let G be a finite Abelian group, and let M and S be the multiplier and splitter sets forming a splitting of G. We say the splitting is nonsingular if $\gcd(|G|, m) = 1$ for all $m \in M$. Otherwise, the splitting is called singular. If for any prime p dividing the order of G there is some $m \in M$ such that $p|m$, then the splitting is called purely singular.

Given a finite $M \subseteq \mathbb{Z}$ and some prime $p \in \mathbb{N}$, we denote by $\delta_p(M)$ the number of elements of M divisible by p. The following is an adaptation of [18, p. 75, Corollary 2] for quasi-crosses, which is required for Theorem 15.

Lemma 14: Let $M = [-k_-, k_+]^*$ be the multiplier set of the (k_+, k_-, n)-quasi-cross. Assume M and S are a purely singular splitting of a finite Abelian group G. Then $\delta_p(M) \geqslant |M|/p^2$ for any prime divisor p of $|G|$.

Proof: Since the splitting is nonsingular, for any prime divisor p of $|G|$, p divides some $m \in M = [-k_-, k_+]^*$. Necessarily, $p \leqslant k_+$. Let us assume

$$k_- = q_- p + r_- \qquad k_+ = q_+ p + r_+$$

where $0 \leqslant r_-, r_+ < p$. We would like, therefore, to prove that

$$\delta_p(M) = q_+ + q_- \geqslant \frac{k_+ + k_-}{p^2}.$$

After rearranging, this is equivalent to proving that

$$pq_+ + pq_- \geqslant \frac{r_+ + r_-}{p-1}.$$

This obviously holds since $p \geqslant 2$, $q_+ \geqslant 1$, and $r_+, r_- \leqslant p-1$, so

$$pq_+ + pq_- \geqslant 2 \geqslant \frac{r_+ + r_-}{p-1}$$

proving the claim. ■

Having proved Lemma 14, the following theorem from [18] directly follows with the exact same proof.

Theorem 15 [18, p. 75, Theorem 9]: Let $M = [-k_-, k_+]^*$ be the multiplier set of the (k_+, k_-, n)-quasi-cross. If M splits G, then M splits $\mathbb{Z}_{|G|}$.

Theorem 15 is important since now, to show the existence or nonexistence of a lattice tiling by (k_+, k_-, n)-quasi-crosses, it is sufficient to check splittings of $\mathbb{Z}_n$. We shall now do exactly that, and reach the conclusion that $(2,1,n)$-quasi-crosses lattice-tile $\mathbb{R}^n$ only with the parameters of Construction 3.

Theorem 16: Let $M = [-(k-1), k]^*$ be the multiplier set of the $(k, k-1, n)$-quasi-cross, $k \geqslant 2$. If M splits a finite Abelian group G, $|G| > 1$, then $\gcd(k, |G|) \neq 1$.

Proof: By Theorem 15 we may assume $G = \mathbb{Z}_q$. Denote the splitter set $S = \{s_1, s_2, \dots, s_n\}$. It is easily seen that if $\gcd(\ell, q) = 1$, then ℓS is also a splitter set. Since $1 = ms$ for some $m \in M$ and $s \in S$, then $\gcd(m, q) = 1$ and $1 \in mS$. We can therefore assume, w.l.o.g., that $s_1 = 1 \in S$.

Since M and S split $\mathbb{Z}_q$, then $q \geqslant 2k$. If $q = 2k$ the claim of the theorem trivially holds. Assume then that $q > 2k$. Let us consider the unique factorization of $-k = ms_i$, $m \in M$ and $s_i \in S$. We note that if $q > 2k$, then $-k \not\equiv m \pmod{q}$ for all $m \in M$, and so $s_i \neq s_1$.

If $-(k-1) \leqslant m \leqslant k-1$, then $-m \in M$ as well, and so

$$k = -ms_i = ks_1,$$

and since $k \in M$, we get a contradiction to the splitting.

The only remaining option is that $m = k$, and $-k = ks_i$. If we assume to the contrary that $\gcd(k, q) = 1$, then we can divide by k and get $s_i = -1$. But then

$$-1 = 1 \cdot s_i = (-1) \cdot s_1,$$

where $1, -1 \in M$, and we get a contradiction to the splitting again. It follows that $\gcd(k, q) \neq 1$. ■

Corollary 17: There is no nonsingular splitting of $\mathbb{Z}_q$ by $M = [-(k-1), k]^*$.

Proof: Assume such a splitting exists, then $\gcd(q, m) = 1$ for all $m \in M$, and in particular $\gcd(q, k) = 1$, contradicting Theorem 16. ■

Theorem 18: Let $M = [-2^w + 1, 2^w]^*$ be the multiplier set of the $(2^w, 2^w - 1, n)$-quasi-cross, $w \in \mathbb{N}$. If M splits $\mathbb{Z}_q$ then $q = 2^{r(w+1)}$ for some $r \in \mathbb{N}$.

Proof: By Theorem 16 and Corollary 17, M cannot split $\mathbb{Z}_q$ nonsingularly and $\gcd(q, 2^w) \neq 1$, i.e., q is even. Denote $q = t2^{r'}$, with $t, r' \in \mathbb{N}$, t odd.

Let S be the splitter set. Because of the splitting, every odd number in $\mathbb{Z}_q$ is represented uniquely as ms, $m \in M$, $s \in S$, where m and s are odd. There are 2^w odd numbers in M and $t2^{r'-1}$ odd numbers in $\mathbb{Z}_q$, so $2^w | t2^{r'-1}$ implying $r' \geqslant w+1$ and the existence of exactly $t2^{r'-(w+1)}$ odd numbers in S.

Multiplying the odd numbers in S by the elements of M covers exactly 2^{w-i} numbers in $\mathbb{Z}_q$ having a residue of 2^i modulo 2^{i+1}, for all $0 \leqslant i \leqslant w$. The only, thus far, uncovered numbers in $\mathbb{Z}_q$ are those having 0 residue modulo 2^{w+1}. These form a group isomorphic to $\mathbb{Z}_{q/2^{w+1}}$. We also conclude that all even numbers in S leave a residue of 0 modulo 2^{w+1}.

We can therefore take $\mathbb{Z}_{q/2^{w+1}}$ and all the even numbers of S divided by 2^{w+1} and repeat the argument above. We conclude $q = t2^{r(w+1)}$ for some $r \in \mathbb{N}$. Also, the repetition of the above argument repeatedly divides q by 2^{w+1}, and stops when we reach the fact that M splits $\mathbb{Z}_t$, t odd. This is impossible by Theorem 16 unless $t = 1$, which completes the proof. ■

As a special case of the above theorems, we reach the following claim.

Corollary 19: The $(2,1,n)$-quasi-cross lattice-tiles $\mathbb{R}^n$ only with the parameters of Construction 3.

Proof: Simply apply Theorem 18 with $w = 1$ and compare with the parameters of Construction 3. ■

V. Conclusion

Motivated by coding for flash memories, we considered lattice tilings of $\mathbb{R}^n$ by (k_+, k_-, n)-quasi-crosses. These lattices form perfect codes correcting a single error with limited magnitudes k_+ and k_- for positive and negative errors, respectively. We showed these lattice tilings are equivalent to certain group splittings, and in certain cases (when the group is cyclic), to modular B_1 sequences.

We provided two constructions which may be used recursively to build infinite families of such lattice tilings for any given rational balance ration $\beta = k_-/k_+$. We also specifically constructed an infinite family of lattice tilings for the $(2,1,n)$-quasi-cross.

We followed by studying bounds on the parameters of such lattice tilings expressed in terms of k_+, k_-, and n. We also examined restrictions on group splitting, and concluded through a special case of the theorems presented, that $(2,1,n)$-quasi-crosses lattice-tile $\mathbb{R}^n$ only with the parameters of the construction presented earlier.

We conclude with the following comment. While Constructions 1 and 2 are quite general, Construction 3 appears to arbitrarily fix $k_+ = 2$ and $k_- = 1$. This raises the question of whether Construction 3, in particular, may be generalized, and in general, whether there exist tilings by quasi-crosses with other parameters. We performed a computer search looking for lattice tilings by (k_+, k_-, n)-quasi-crosses. It was found that for all $0 < k_- < k_+ \leqslant 10$ and split group $G = \mathbb{Z}_q$ of order $q \leqslant 100$, that only lattice tilings with the parameters of the constructions provided in this paper exist.

Acknowledgment

The author would like to thank the two anonymous reviewers, whose comments improved the presentation of this paper.

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