Two-Dimensional Cluster-Correcting Codes

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***Abstract*—We consider two-dimensional error-correcting codes capable of correcting a single arbitrary cluster of errors of size b. We provide optimal 2-cluster-correcting codes in several connectivity models, as well as optimal, or nearly optimal, 2-cluster-correcting codes in all dimensions. We also construct 3-cluster-correcting codes and b-straight-cluster-correcting codes. We conclude by improving the Reiger bound for two-dimensional cluster-correcting codes.**

***Index Terms*—Burst-correcting codes, cluster-correcting codes, two-dimensional codes.**

I. Introduction

CURRENT memory devices require that information is stored on two-dimensional surfaces. Up until now, one-dimensional codes have been used for such applications by folding the one-dimensional data into the two-dimensional surface. The main disadvantage of this approach is that the devices are not capable of handling "real" two-dimensional error patterns. More recent developments in technology require that two-dimensional error patterns are recovered. These applications include optical recordings such as page-oriented optical memories [19], and volume holographic storage [13], [14].

The difference between one-dimensional and two-dimensional error-correcting codes comes to light when we consider *burst errors*, also called *cluster errors*. In one dimension, we say that a burst error of length b occurred if all the errors are confined to b consecutive positions. In two dimensions, several models of cluster errors may be found in the literature. Most two-dimensional codes are designed to correct a single cluster error of a given rectangular shape, say a $b_1 \times b_2$ rectangular array [1], [4], [8], [11], [15], [16]. Other works consider the rank of the error array [12], [21], or criss-cross patterns [21], [5], [22].

In some recent papers [3], [6], [10], [24], [18], [17], it is assumed that a cluster error can have an arbitrary shape. The approach in these papers is to use interleaving schemes. The method is very efficient as it uses one-dimensional error-correcting codes with two-dimensional interleaving, but the redundancy of the constructed codes can be considerably improved. This is because the interleaving schemes proposed can correct other types of errors in addition to clusters. In fact, they can correct any dispersed pattern of errors as long as not too many errors occur within one of its subcodes.

In this paper, we construct error-correcting codes capable of correcting a single cluster error with arbitrary shape by using a direct algebraic approach. We define connectivity models for the two-dimensional surface. A cluster error of size b is a set of b points on the surface such that between any two points of the cluster, there is path consisting of points from the cluster. A code is called a b-cluster-correcting code if it is capable of correcting any number of errors which occur in a single cluster of size b. The reader should note the difference in naming conventions from random-error-correcting codes. While an e-error-correcting code is capable of correcting up to e different errors, a b-cluster-correcting code is capable of correcting just *one* cluster, whose size is at most b. We design codes which are capable of correcting small cluster errors. We also prove lower bounds on the redundancy of cluster-correcting codes.

The rest of this paper is organized as follows. In Section II, we introduce basic definitions of linear two-dimensional codes and three connectivity models for two-dimensional surfaces. In Section III, we construct optimal two-dimensional 2-cluster-correcting codes in the three connectivity models and generalize one of the constructions to higher dimensions. In Section IV, we construct 3-cluster-correcting codes, and in Section V, we restrict our discussion to two-dimensional clusters on straight lines. In Section VI, we improve the well-known Reiger bound for one-dimensional burst-correcting codes to two-dimensional cluster-correcting codes. Conclusions and problems for further research are given in Section VII.

II. Basic Definitions

A two-dimensional linear code C is a linear subspace of the $n_1 \times n_2$ binary matrices. If the subspace is of dimension $n_1 n_2 - r$, we say that the code is an $[n_1 \times n_2, n_1 n_2 - r]$ code. The code is also defined by its parity-check matrix. Let $H = (h_{ijk})$ be an $n_1 \times n_2 \times r$ three-dimensional binary matrix, consisting of r linearly independent $n_1 \times n_2$ matrices, and let $c = (c_{ij})$ denote a binary $n_1 \times n_2$ matrix. The linear subspace defined by the following set of r equations

$$\sum_{i=0}^{n_1-1} \sum_{j=0}^{n_2-1} c_{ij} h_{ijk} = 0$$

for all $0 \leq k \leq r-1$, is an $[n_1 \times n_2, n_1 n_2 - r]$ code. We say that r is the redundancy of the code.

The first connectivity model is called the $+$ model. In this model, a point (i,j) has the following four neighbors:

$$\{(i+1,j),(i-1,j),(i,j+1),(i,j-1)\}.$$

Manuscript received October 15, 2004; revised February 2, 2005. The material in this paper was presented in part in the IEEE International Symposium on Information Theory, Chicago, IL, June/July 2004.

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Communicated by K. A. S. Abdel-Ghaffar, Associate Editor for Coding Theory.

Digital Object Identifier 10.1109/TIT.2005.847726

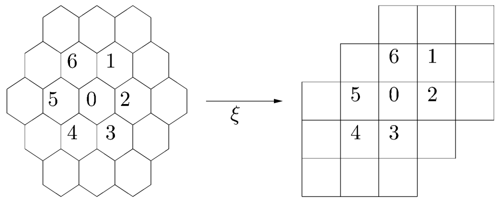

Fig. 1. The $*$ model translation.

When (i,j) is an edge point, the neighbor set is reduced to points within the array. Unless specified otherwise, this is the default model we use throughout the paper.

The second model is called the $\circledast$ model, in which each point (i,j) has eight neighbors

$$\{(i+1,j),(i-1,j),(i,j+1),(i,j-1),\\ (i+1,j+1),(i-1,j+1),(i+1,j-1),(i-1,j-1)\}.$$

The last model is called the $*$ model. Instead of the rectangular grid we have used up to now, we define the following graph. We start by tiling the plane $\mathbb{R}^2$ with regular hexagons. The vertices of the graph are the center points of the hexagons. We connect two vertices if and only if their respective hexagons are adjacent. This way, each vertex has exactly six neighboring vertices.

We will use an isomorphic representation of the model. This representation includes $\mathbb{Z}^2$ as the set of vertices. Each point $(x,y)\in\mathbb{Z}^2$ has the following neighboring vertices:

$$\{(x+a,y+b)\mid a,b\in\{-1,0,1\},a+b\neq 0\}.$$

It may be shown that the two models are isomorphic by using the mapping $\xi:\mathbb{R}^2\to\mathbb{Z}^2$, which is defined by

$$\xi(x,y)=\left(\frac{x}{\sqrt{3}}+\frac{y}{3},\frac{2y}{3}\right).$$

The effect of the mapping on the neighbor set is shown in Fig. 1. From now on, by abuse of notation, we will also call the last model—the $*$ model.

Traditionally, the rows and columns of arrays are indexed in ascending order, top to bottom and left to right. As far as rows are concerned, this is a mirror image to the numbering scheme of $\mathbb{Z}^2$. In the case of the $*$ model, we arbitrarily choose to take the neighbors of point (i,j) to be

$$\{(i+1,j),(i-1,j),(i,j+1),(i,j-1),\\ (i-1,j+1),(i+1,j-1)\}.$$

All the neighbor sets of the different models are summarized in Fig. 2. A square with a dot is point (i,j). These three models were also considered in [26], in the discussion on two-dimensional constrained codes, where they are called diamond, square, and hexagonal, instead of $+$, $\circledast$, and $*$, respectively.

III. 2-Cluster-Correcting Codes

In this section, we present constructions for optimal 2-cluster-correcting codes. We consider two-dimensional codes in all three connectivity models and multidimensional codes in the generalization of the $+$ model. All codes are presented by their parity-check matrix. We prove that the codes are indeed 2-cluster-correcting codes by showing that all the relevant syndromes are distinct. Decoding procedures can be easily derived from the structure of these syndromes.

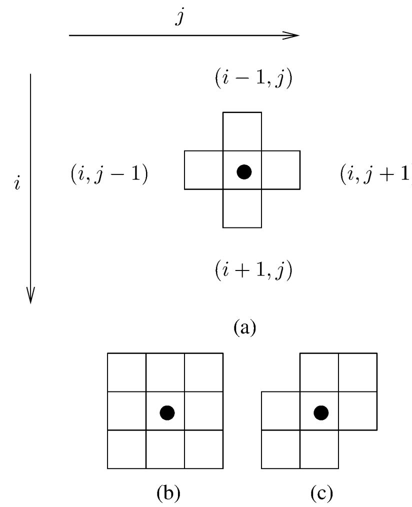

Fig. 2. Neighbors of position (i,j) in (a) the $+$ model, (b) the $\circledast$ model, (c) the $*$ model. The row index i and column index j increase in the direction of the arrows.

A. Two-Dimensional Codes

1) The $+$ Model:

Construction A: Let α be a primitive element of $\mathrm{GF}(2^m)$, $m\geq 2$. We construct the following $(2^m-1)\times(2^m-1)\times(2m+2)$ parity-check matrix $H_2(m)$:

$$h_{ij}=\begin{bmatrix}1\\ i \bmod 2\\ \alpha^{i+j}\\ \alpha^{i-j}\end{bmatrix},\qquad \text{for all } 0\leq i,j\leq 2^m-2.$$

Theorem 1: $H_2(m)$ is a parity-check matrix of a 2-cluster-correcting code in the $+$ model.

Proof: There are three types of errors that the code should correct: an error in a single position, errors in two horizontally adjacent positions, and errors in two vertically adjacent positions. These errors result in the following three types of syndromes, respectively:

$$\begin{bmatrix}1\\ i \bmod 2\\ \alpha^{i+j}\\ \alpha^{i-j}\end{bmatrix},\quad \begin{bmatrix}0\\ 0\\ (1+\alpha)\alpha^{i+j}\\ (1+\alpha)\alpha^{i-j-1}\end{bmatrix},\quad \begin{bmatrix}0\\ 1\\ (1+\alpha)\alpha^{i+j}\\ (1+\alpha)\alpha^{i-j+1}\end{bmatrix}.$$

Therefore, given a syndrome it is clear which type of error occurred. Moreover, it can be readily verified that any two syndromes of the same type are distinct. $\square$

We note that behind Construction A there is a simple intuition. After calculating the syndrome of the received word and finding it to be nonzero, the top bit determines whether we have one or two positions in error. The next bit gives us the direction (horizontal or vertical) of the error when there are two positions in error. Finally, the two lower layers of m bits each determine the exact position of the error. This flavor of construction will be used throughout the paper: some of the syndrome bits determine the pattern of the error, while others locate it. This is further demonstrated in the following example.

Example 1: Let us take $H_2(3)$ as a parity-check matrix of the 2-cluster-correcting code of size 7×7. Suppose we get the following syndrome:

$$s = \begin{bmatrix} 0 \\ 1 \\ \alpha^5 \\ \alpha^2 \end{bmatrix}$$

where α is a primitive element of GF (8). Since the syndrome is nonzero, we start the decoding process by identifying the burst pattern. The top bit is "0," which means that a cluster error of size 2 occurred. The next bit is "1," hence, the error occurred along a column. We now have to locate the cluster error by solving the set of equations

$$\begin{aligned} \alpha^{i+j} + \alpha^{i+j+1} &= \alpha^5 \\ \alpha^{i-j} + \alpha^{i-j+1} &= \alpha^2 \end{aligned}$$

which reduces to a set of two equations modulo 7. The solution is $i = 4$ and $j = 5$.

The code defined by $H_2(m)$ has redundancy $2m + 2$. The total number of possible different 2-cluster errors in a $(2^m - 1) \times (2^m - 1)$ array is $3(2^m - 1)^2 - 2(2^m - 1)$. Hence, a lower bound on the redundancy of a two-dimensional code of size $(2^m - 1) \times (2^m - 1)$ is

$$\left\lceil \log_2 \left(3(2^m - 1)^2 - 2(2^m - 1) + 1\right) \right\rceil = 2m + 2$$

for $m \geq 3$.

Corollary 1: $H_2(m)$ is a parity-check matrix of an optimal 2-cluster-correcting code in the $+$ model, for all $m \geq 3$.

2) The $\ast$ Model: The following lemma from [2] is useful in some of the constructions which follow.

Lemma 1: There exists a primitive element α in GF (2^m), m even, $m \geq 4$, such that $\log_\alpha(1 + \alpha) \not\equiv 2 \pmod 3$.

Construction B: Let α be a primitive element of GF (2^m), m even, $\log_\alpha(1 + \alpha) \not\equiv 2 \pmod 3$, and β a primitive element of GF (4). We construct the following $(2^m - 1) \times (2^m - 1) \times (2m + 3)$ parity-check matrix $H_2^{\ast}(m)$:

$$h_{ij} = \begin{bmatrix} j \bmod 2 \\ \beta^{i+2j} \\ \alpha^{i+2j} \\ \alpha^{i-2j} \end{bmatrix}, \qquad \text{for all } 0 \leq i,\ j \leq 2^m - 2.$$

Theorem 2: $H_2^{\ast}(m)$ is a parity-check matrix of a 2-cluster-correcting code in the $\ast$ model.

Proof: Again, we show that all possible syndromes are distinct. There are five different types of 2-cluster errors as depicted in Fig. 3. Let $s_k(i, j)$ denote the syndrome of type k, $1 \leq k \leq 5$, with its reference point positioned at (i, j). The reference points are the squares with a dot. Each square represents a position in error. We therefore have the following five possible syndromes:

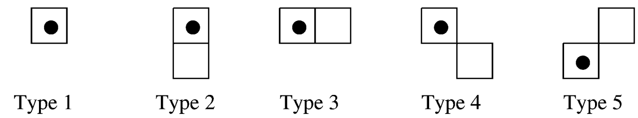

Fig. 3. 2-burst error types in the $\ast$ model.

$$s_1(i,j) = \begin{bmatrix} j \bmod 2 \\ \beta^{i+2j} \\ \alpha^{i+2j} \\ \alpha^{i-2j} \end{bmatrix}$$

$$s_2(i,j) = \begin{bmatrix} 0 \\ \beta^{i+2j+2} \\ (1+\alpha)\alpha^{i+2j} \\ (1+\alpha)\alpha^{i-2j} \end{bmatrix}$$

$$s_3(i,j) = \begin{bmatrix} 1 \\ \beta^{i+2j+1} \\ (1+\alpha^2)\alpha^{i+2j} \\ (1+\alpha^2)\alpha^{i-2j-2} \end{bmatrix}$$

$$s_4(i,j) = \begin{bmatrix} 1 \\ 0 \\ (1+\alpha^3)\alpha^{i+2j} \\ (1+\alpha)\alpha^{i-2j-1} \end{bmatrix}$$

$$s_5(i,j) = \begin{bmatrix} 1 \\ \beta^{i+2j+2} \\ (1+\alpha)\alpha^{i+2j} \\ (1+\alpha^3)\alpha^{i-2j-3} \end{bmatrix}.$$

Two distinct cluster errors of the same type have different syndromes. This is shown by examining the exponents of α. For example, assume $s_5(i, j) = s_5(i', j')$. This gives us

$$\begin{aligned} (1+\alpha)\alpha^{i+2j} &= (1+\alpha)\alpha^{i'+2j'} \\ (1+\alpha^3)\alpha^{i-2j-3} &= (1+\alpha^3)\alpha^{i'-2j'-3}. \end{aligned}$$

After rearranging we obtain

$$\begin{aligned} i + 2j &\equiv i' + 2j' \pmod{2^m - 1} \\ i - 2j &\equiv i' - 2j' \pmod{2^m - 1}. \end{aligned}$$

Since $2^m - 1$ is odd, division by 2 is well defined and we have

$$i \equiv i' \pmod{2^m - 1} \quad j \equiv j' \pmod{2^m - 1}$$

but $0 \leq i, i', j, j' \leq 2^m - 2$, and hence, $i = i'$ and $j = j'$.

Finally, we consider pairs of syndromes of different types. Six pairs of types are easily seen to be distinct. For the other four pairs, let $k = \log_\alpha(1+\alpha)$. We write equations on the exponents of α and β. Since the orders of α and β are divisible by 3, we can reduce the equations modulo 3.

Types 1 and 2: Assume $s_1(i, j) = s_2(i', j')$. Hence,

$$\begin{aligned} i + 2j &\equiv i' + 2j' + 2 \pmod 3 \\ i + 2j &\equiv i' + 2j' + k \pmod 3. \end{aligned}$$

Types 1 and 3: Assume $s_1(i, j) = s_3(i', j')$. Hence,

$$\begin{aligned} i + 2j &\equiv i' + 2j' + 1 \pmod 3 \\ i + 2j &\equiv i' + 2j' + 2k \pmod 3. \end{aligned}$$

Types 1 and 5: Assume $s_1(i,j) = s_5(i',j')$. Hence,

$$\begin{aligned} i+2j &\equiv i'+2j'+2 \pmod 3 \\ i+2j &\equiv i'+2j'+k \pmod 3. \end{aligned}$$

Types 3 and 5: Assume $s_3(i,j) = s_5(i',j')$. Then,

$$\begin{aligned} i+2j+1 &\equiv i'+2j'+2 \pmod 3 \\ i+2j+2k &\equiv i'+2j'+k \pmod 3. \end{aligned}$$

In all four cases, the two equations imply that $k \equiv 2 \pmod 3$ which is a contradiction. Hence, each pair has two distinct syndromes. □

The code defined by $H_2^{\circledast}(m)$ has redundancy $2m+3$. The total number of possible different 2-cluster errors in a $(2^m-1)\times(2^m-1)$ array is $5(2^m-1)^2 - 6(2^m-1)+2$. Hence, a lower bound on the redundancy of a two-dimensional code of size $(2^m-1)\times(2^m-1)$ is

$$\left\lceil \log_2\left(5(2^m-1)^2 - 6(2^m-1)+3\right)\right\rceil = 2m+3$$

for all $m \geq 5$.

Corollary 2: $H_2^{\circledast}(m)$ is a parity-check matrix of an optimal 2-cluster-correcting code in the $\circledast$ model, for all even $m \geq 6$.

For odd $m \geq 3$, we can design a $(2^m-1)\times(2^m-1)\times(2m+4)$ parity-check matrix for a 2-cluster-correcting code. The definition of the columns of the parity-check matrix is given by

$$h_{ij} = \begin{bmatrix} 1 \\ i \bmod 2 \\ j \bmod 2 \\ \lfloor \frac{i+j}{2} \rfloor \bmod 2 \\ \alpha^{i+2j} \\ \alpha^{i-2j} \end{bmatrix}, \qquad \text{for all } 0 \leq i,\ j \leq 2^m-2.$$

Since by the previous calculation we need redundancy of at least $2m+3$ for $m \geq 5$, this code is nearly optimal, having one redundancy bit above the sphere-packing bound.

3) The $*$ Model: We now give a similar construction for the $*$ model.

Construction C: Let α be a primitive element of GF(2^m), m even, $\log_\alpha(1+\alpha) \not\equiv 2 \pmod 3$, and β a primitive element of GF(4). We construct the following $(2^m-1)\times(2^m-1)\times(2m+2)$ parity-check matrix $H_2^*(m)$:

$$h_{ij} = \begin{bmatrix} \beta^{i-2j} \\ \alpha^{i+2j} \\ \alpha^{i-2j} \end{bmatrix}, \qquad \text{for all } 0 \leq i,\ j \leq 2^m-2.$$

Theorem 3: $H_2^*(m)$ is a parity-check matrix of a 2-cluster-correcting code in the $*$ model.

Proof: Again, we show that all possible syndromes are distinct. There are four different types of 2-cluster errors as depicted in Fig. 4. Let $s_k(i,j)$ denote the syndrome of type k, $1 \leq k \leq 4$, with its reference point positioned at (i,j). The reference points are the squares with a dot. Each square represents a position in error. We therefore have the following four possible syndromes:

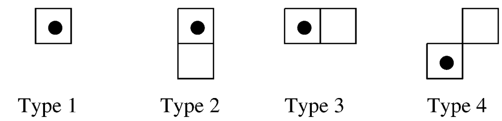

Fig. 4. 2-cluster errors types in the $*$ model.

$$s_1(i,j) = \begin{bmatrix} \beta^{i-2j} \\ \alpha^{i+2j} \\ \alpha^{i-2j} \end{bmatrix}$$

$$s_2(i,j) = \begin{bmatrix} \beta^{i-2j+2} \\ (1+\alpha)\alpha^{i+2j} \\ (1+\alpha)\alpha^{i-2j} \end{bmatrix}$$

$$s_3(i,j) = \begin{bmatrix} \beta^{i-2j+2} \\ (1+\alpha^2)\alpha^{i+2j} \\ (1+\alpha^2)\alpha^{i-2j-2} \end{bmatrix}$$

$$s_4(i,j) = \begin{bmatrix} 0 \\ (1+\alpha)\alpha^{i+2j} \\ (1+\alpha^3)\alpha^{i-2j-3} \end{bmatrix}.$$

The rest of the proof is similar to the proof of Theorem 2 and hence it is omitted. □

The code defined by $H_2^*(m)$ has redundancy $2m+2$. The total number of possible different 2-cluster errors in the array is $(2^{m+1}-3)^2$. Hence, a lower bound on the redundancy of a two-dimensional code of size $(2^m-1)\times(2^m-1)$ is

$$\left\lceil \log_2\left((2^{m+1}-3)^2+1\right)\right\rceil = 2m+2,$$

for $m \geq 3$.

Corollary 3: H_2^* is a parity-check matrix of an optimal 2-cluster-correcting code in the $*$ model, for all even $m \geq 4$.

For odd m we can design a $(2^m-1)\times(2^m-1)\times(2m+3)$ parity-check matrix for a 2-cluster-correcting code. The definition of the columns of the parity-check matrix is given by

$$h_{ij} = \begin{bmatrix} 1 \\ i \bmod 2 \\ j \bmod 2 \\ \alpha^{i+2j} \\ \alpha^{i-2j} \end{bmatrix}, \qquad \text{for all } 0 \leq i,\ j \leq 2^m-2.$$

Again, this code is nearly optimal for $m \geq 3$, having one redundancy bit above the sphere-packing bound.

B. Multidimensional Codes

The construction of Section III-A1 for the $+$ model can be generalized to higher dimensions. We also have to generalize the $+$ model, and we do so in the obvious way: for the D-dimensional case, each point has $2D$ neighbors which are the two adjacent points along each of the D dimensions.

Construction D: Let $D \geq 2$ be an integer, $d = \lceil \log_2 D \rceil$, and let α be a primitive element of GF(2^m), $m \geq 2$. Denote

$\boldsymbol{i} = (i_0, i_1, \ldots, i_{D-1})$, where $0 \le i_j \le 2^m - 2$ for all $0 \le j \le D-1$. Let A be a $d \times D$ matrix containing distinct binary d-tuples as columns, and let $B = (b_{ij})$ be the $D \times D$ matrix defined by

$$b_{ij} = \begin{cases} 1, & \text{if } i \ge j \\ -1, & \text{otherwise.} \end{cases}$$

By abuse of notation, if $\boldsymbol{v} = (v_0, \ldots, v_k)^T$ is a column vector, we denote $\alpha^{\boldsymbol{v}} = (\alpha^{v_0}, \ldots, \alpha^{v_k})^T$. We construct the following $(\times(2^m-1))^D \times (Dm + d + 1)$ parity-check matrix $H_2^D(m)$:

$$h_{\boldsymbol{i}} = \begin{bmatrix} 1 \\ A\boldsymbol{i}^T \bmod 2 \\ \alpha^{B\boldsymbol{i}^T} \end{bmatrix}$$

for all $\boldsymbol{i} = (i_0, i_1, \ldots, i_{D-1})$, $0 \le i_j \le 2^m - 2$, where by $(\times(2^m-1))^D$ we mean the product of D terms of $2^m - 1$.

Theorem 4: $H_2^D(m)$ is a parity-check matrix of a D-dimensional 2-cluster-correcting code.

Proof: We show that different 2-cluster errors produce different syndromes. Obviously, the first bit of the syndrome allows us to distinguish between one and two errors.

We start by considering syndromes which result from a single error. Let

$$\begin{bmatrix} 1 \\ u^T \\ \alpha^{v^T} \end{bmatrix}$$

be a syndrome resulting from a single error. The set of equations $B\boldsymbol{i}^T \equiv \boldsymbol{v}^T \pmod{2^m - 1}$ has a unique solution (up to modulo $2^m - 1$)

$$i_j \equiv \frac{v_j - v_{j+1}}{2} \pmod{2^m - 1}, \qquad \text{for } 0 \le j \le D-2$$

$$i_{D-1} \equiv \frac{v_0 + v_{D-1}}{2} \pmod{2^m - 1}.$$

We note that division by 2 is well defined since $2^m - 1$ is odd. Therefore, two syndromes which result from exactly one error, in two different positions, are distinct.

Assume we have two adjacent errors along the jth dimension. Hence, the first error is in position $\boldsymbol{i} = (i_0, i_1, \ldots, i_{D-1})$, and the second error is in position $\boldsymbol{i} + \boldsymbol{e}_j$, where $\boldsymbol{e}_j$ is a vector of weight one with a $\mathbf{1}$ in the jth entry. Since

$$A\boldsymbol{i}^T + A(\boldsymbol{i} + \boldsymbol{e}_j)^T \equiv A\boldsymbol{e}_j^T \pmod 2$$

which is simply the jth column of A, it follows that two adjacent errors along different dimensions have distinct syndromes. Hence, we only have to consider pairs of two adjacent errors along the same dimension.

Let $\boldsymbol{i} = (i_0, i_1, \ldots, i_{D-1})$ and $\boldsymbol{i}' = (i'_0, i'_1, \ldots, i'_{D-1})$ be two distinct positions. Assume that the syndromes of the two pairs of adjacent errors in positions $\{\boldsymbol{i}, \boldsymbol{i} + \boldsymbol{e}_j\}$ and $\{\boldsymbol{i}', \boldsymbol{i}' + \boldsymbol{e}_j\}$ have the same syndromes. Hence,

$$\alpha^{B\boldsymbol{i}^T} + \alpha^{B(\boldsymbol{i}+\boldsymbol{e}_j)^T} = \alpha^{B\boldsymbol{i}'^T} + \alpha^{B(\boldsymbol{i}'+\boldsymbol{e}_j)^T}.$$

Therefore,

$$\alpha^{B\boldsymbol{i}^T} \odot (\mathbf{1} + \alpha^{B\boldsymbol{e}_j^T}) = \alpha^{B\boldsymbol{i}'^T} \odot (\mathbf{1} + \alpha^{B\boldsymbol{e}_j^T})$$

where $\odot$ is componentwise multiplication, and $\mathbf{1}$ is the all-ones vector. Since there is no zero entry in $B\boldsymbol{e}_j^T$, it follows that there is no zero entry in $\mathbf{1} + \alpha^{B\boldsymbol{e}_j^T}$ and hence we have

$$\alpha^{B\boldsymbol{i}^T} = \alpha^{B\boldsymbol{i}'^T}.$$

But, we already showed that for a given vector $\boldsymbol{v}$, the set of equations $B\boldsymbol{i}^T \equiv \boldsymbol{v}^T \pmod{2^m - 1}$ has a unique solution (up to modulo $2^m - 1$). Therefore, $\boldsymbol{i} = \boldsymbol{i}'$, contradicting our assumption. Thus, any pair of two adjacent errors along the same dimension has a distinct syndrome. □

Note that Construction A is obtained from Construction D when $D = 2$. The generalization is quite simple: we still have one bit determining whether one or two errors occurred, we extend the one bit of Construction A used for burst-direction detection to d bits because of the added dimensions, and we extend the $2m$ error-locating bits of Construction A to Dm bits.

For any given dimension $D \ge 2$ we obtain a 2-cluster-correcting code with codewords of size $(\times(2^m-1))^D$ and redundancy $Dm + \lceil \log_2 D \rceil + 1$. The total number of cluster errors in the array is given by $(D+1)(2^m-1)^D - D(2^m-1)^{D-1}$. Hence, a lower bound on the redundancy of a multidimensional code of size $(\times(2^m-1))^D$ is

$$\left\lceil \log_2 \left((D+1)(2^m-1)^D - D(2^m-1)^{D-1} + 1 \right) \right\rceil.$$

If D is a power of 2 and m is large enough then

$$\left\lceil \log_2 \left((D+1)(2^m-1)^D - D(2^m-1)^{D-1} + 1 \right) \right\rceil = \\ = Dm + \log_2 D + 1$$

and therefore we have the following corollary.

Corollary 4: $H_2^D(m)$ is a parity-check matrix of a D-dimensional 2-cluster-correcting code in the $+$ model which, for m large enough, is optimal when D is a power of 2, and otherwise, with one redundancy bit above the lower bound.

IV. 3-Cluster-Correcting Codes

In this section, we present a construction, similar to the constructions of Section III, which corrects 3-cluster errors in the $+$ model.

Construction E: Let α be a primitive element of GF(2^m), m even, $\log_\alpha(1+\alpha) \not\equiv 2 \pmod 3$, and β a primitive element of GF(4). We construct the following $(2^m-1) \times (2^m-1) \times (2m+7)$ parity-check matrix $H_3(m)$:

$$h_{ij} = \begin{bmatrix} 1 \\ \beta^i \\ \beta^{i+2j} \\ \beta^{i-2j} \\ \alpha^{i+2j} \\ \alpha^{i-2j} \end{bmatrix}, \qquad \text{for all } 0 \le i,\ j \le 2^m - 2.$$

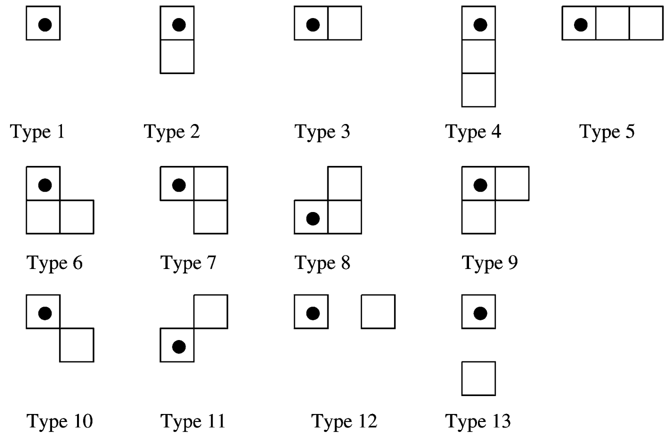

Fig. 5. 3-cluster errors types.

Theorem 5: $H_3(m)$ is a parity-check matrix of a 3-cluster-correcting code.

Proof: We proceed to show that the syndromes of distinct burst errors are distinct. For convenience, let $s_k(i,j)$ denote the syndrome of type k, $1 \leq k \leq 13$, with its reference point positioned at (i,j). The 13 different types of syndromes of 3-cluster errors are depicted in Fig. 5. The reference points are the squares with a dot. Each square represents a position in error. We therefore have the following thirteen possible syndromes:

$$s_1(i,j) = \begin{bmatrix} 1 \\ \beta^i \\ \beta^{i+2j} \\ \beta^{i-2j} \\ \alpha^{i+2j} \\ \alpha^{i-2j} \end{bmatrix}$$

$$s_2(i,j) = \begin{bmatrix} 0 \\ \beta^{i+2} \\ \beta^{i+2j+2} \\ \beta^{i-2j+2} \\ (1+\alpha)\alpha^{i+2j} \\ (1+\alpha)\alpha^{i-2j} \end{bmatrix}$$

$$s_3(i,j) = \begin{bmatrix} 0 \\ 0 \\ \beta^{i+2j+1} \\ \beta^{i-2j-1} \\ (1+\alpha^2)\alpha^{i+2j} \\ (1+\alpha^2)\alpha^{i-2j-2} \end{bmatrix}$$

$$s_4(i,j) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ (1+\alpha+\alpha^2)\alpha^{i+2j} \\ (1+\alpha+\alpha^2)\alpha^{i-2j} \end{bmatrix}$$

$$s_5(i,j) = \begin{bmatrix} 1 \\ \beta^i \\ 0 \\ 0 \\ (1+\alpha^2+\alpha^4)\alpha^{i+2j} \\ (1+\alpha^2+\alpha^4)\alpha^{i-2j-4} \end{bmatrix}$$

$$s_6(i,j) = \begin{bmatrix} 1 \\ \beta^i \\ \beta^{i+2j+1} \\ 0 \\ (1+\alpha+\alpha^3)\alpha^{i+2j} \\ (1+\alpha+\alpha^2)\alpha^{i-2j-1} \end{bmatrix}$$

$$s_7(i,j) = \begin{bmatrix} 1 \\ \beta^{i+1} \\ \beta^{i+2j+2} \\ 0 \\ (1+\alpha^2+\alpha^3)\alpha^{i+2j} \\ (1+\alpha+\alpha^2)\alpha^{i-2j-2} \end{bmatrix}$$

$$s_8(i,j) = \begin{bmatrix} 1 \\ \beta^{i+2} \\ 0 \\ \beta^{i-2j-2} \\ (1+\alpha+\alpha^2)\alpha^{i+2j} \\ (1+\alpha+\alpha^3)\alpha^{i-2j-3} \end{bmatrix}$$

$$s_9(i,j) = \begin{bmatrix} 1 \\ \beta^{i+1} \\ 0 \\ \beta^{i-2j} \\ (1+\alpha+\alpha^2)\alpha^{i+2j} \\ (1+\alpha^2+\alpha^3)\alpha^{i-2j-2} \end{bmatrix}$$

$$s_{10}(i,j) = \begin{bmatrix} 0 \\ \beta^{i+2} \\ 0 \\ \beta^{i-2j+1} \\ (1+\alpha^3)\alpha^{i+2j} \\ (1+\alpha)\alpha^{i-2j-1} \end{bmatrix}$$

$$s_{11}(i,j) = \begin{bmatrix} 0 \\ \beta^{i+1} \\ \beta^{i+2j+2} \\ 0 \\ (1+\alpha)\alpha^{i+2j} \\ (1+\alpha^3)\alpha^{i-2j-3} \end{bmatrix}$$

$$s_{12}(i,j) = \begin{bmatrix} 0 \\ 0 \\ \beta^{i+2j+2} \\ \beta^{i-2j+1} \\ (1+\alpha^4)\alpha^{i+2j} \\ (1+\alpha^4)\alpha^{i-2j-4} \end{bmatrix}$$

$$s_{13}(i,j) = \begin{bmatrix} 0 \\ \beta^{i+1} \\ \beta^{i+2j+1} \\ \beta^{i-2j+1} \\ (1+\alpha^2)\alpha^{i+2j} \\ (1+\alpha^2)\alpha^{i-2j} \end{bmatrix}.$$

The rest of the proof is very similar to the proof of Theorem 2. As in Theorem 2, only four pairs of types of syndromes have to be compared: types 2 and 13, types 3 and 12, types 6 and 7, and types 8 and 9. We leave the completion of the proof to the reader. □

The code defined by $H_3(m)$ has redundancy $2m + 7$. The total number of possible different 3-cluster errors in a $(2^m-1) \times (2^m-1)$ array is $13 \cdot 2^{2m} - 48 \cdot 2^m + 41$. Hence, a lower bound on the redundancy of a two-dimensional code of size $(2^m-1)\times(2^m-1)$ is $2m+4$ for all $m\geq 4$.

V. b-Straight-Cluster-Correcting Codes

A very natural type of a cluster error is one in which the errors are confined to b consecutive positions on the same line. In the $+$ model there are two directions of lines: along a row and along a column. In the $*$ model there are three directions of lines, and in the $\circledast$ model there are four directions. A cluster of errors confined to positions on a line will be called *straight*. In the following three subsections, we describe constructions of codes capable of correcting b-straight-cluster errors in the three models. In the sequel let $\boldsymbol{e}_i$, $1\leq i\leq b-1$, be the vector of length $b-1$, weight one, with *one* in the ith entry. By abuse of notation we define $\boldsymbol{e}_0$ to be the all-zero vector.

A. The $+$ Model

Construction F: Let H be an $(m+b-1)\times(2^m-1)$ parity-check matrix for a one-dimensional cyclic b-burst-correcting code (see [2]), where $m\geq b$. We denote the columns of H by $h_0,h_1,\ldots,h_{2^m-2}$. Let α be a primitive element of GF(2^m). We construct the following $(2^m-1)\times(2^m-1)\times(2m+2b-2)$ parity-check matrix $H_b^{(S)}(m)$:

$$h_{ij}=\begin{bmatrix}\boldsymbol{e}_{i \bmod b}\\ h_{i-j}\\ \alpha^{i+j}\end{bmatrix},\qquad \text{for all } 0\leq i,j\leq 2^m-2$$

where the index of h_{i-j} is taken modulo 2^m-1.

Theorem 6: $H_b^{(S)}(m)$ is a parity-check matrix of a b-straight-cluster-correcting code.

Proof: We prove that all syndromes associated with b-straight-cluster errors are distinct. Let $s=(s_0,s_1,s_2)^T$, where the lengths of s_0, s_1, s_2, are $b-1$, $m+b-1$, and m, respectively, be a syndrome. We use s_0 to determine whether the errors occurred in a row or in a column. If the weight of s_0 is one, either an odd number of errors occurred on a single row or two errors occurred in a single column. To distinguish between the two, we can find the linear combination of columns from H that form s_1^T. If the weight of s_0 is zero then all errors occurred on a single row. If the weight of s_1 is $k>1$ then k or $k+1$ errors occurred in a single column.

We note, that once it is known in which row (column) the errors occurred, the positions of the errors in the row (column) are uniquely determined by s_1, since the appropriate entries of $H_b^{(S)}(m)$ form a cyclic shift of a parity-check matrix of a cyclic b-burst-correcting code.

To complete the proof, we show that if two cluster errors along two different rows (columns) have syndromes

$$s=(s_0,s_1,s_2)^T\quad\text{and}\quad s'=(s_0,s_1,s_3)^T$$

then $s_2\neq s_3$. Assume that the cluster error occurred in a row. Denote the error pattern positions along the row, up to a cyclic shift, by $0=j_0<j_1<\cdots<j_\ell<b$. Now, assume that the first cluster error occurs along row i_1 at positions $J_{i_1}+j_0,J_{i_1}+j_1,\ldots,J_{i_1}+j_\ell$, and the second, along row $i_2>i_1$ at positions $J_{i_2}+j_0,J_{i_2}+j_1,\ldots,J_{i_2}+j_\ell$. Since the first $m+2b-2$ positions in the two syndromes are the same, it follows that

$$i_1-J_{i_1}\equiv i_2-J_{i_2}\pmod{2^m-1}.$$

We also have that

$$s_2=\alpha^{i_1+J_{i_1}}(\alpha^{j_0}+\alpha^{j_1}+\cdots+\alpha^{j_\ell})$$

and

$$s_3=\alpha^{i_2+J_{i_2}}(\alpha^{j_0}+\alpha^{j_1}+\cdots+\alpha^{j_\ell}).$$

Note that since $m\geq b$, we have $\alpha^{j_0}+\alpha^{j_1}+\cdots+\alpha^{j_\ell}\neq 0$. If we assume that $s_2=s_3$ then

$$\begin{aligned}i_1+J_{i_1}&\equiv i_2+J_{i_2}\pmod{2^m-1}\\ i_1-J_{i_1}&\equiv i_2-J_{i_2}\pmod{2^m-1}\end{aligned}$$

which implies that $i_1=i_2$, a contradiction. Thus, $s_2\neq s_3$. A similar argument works when the cluster error occurs in a column. □

The code defined by $H_b^{(S)}(m)$ has redundancy $2m+2b-2$. The total number of possible different b-straight-cluster errors is $2^b(2^m-1)(2^m-b+1)-(2^{2m}-1)$. Hence, a lower bound on the redundancy of a two-dimensional code of size $(2^m-1)\times(2^m-1)$ is $2m+b$, for m large enough. Therefore, Construction F produces codes with $b-2$ redundancy bits above the lower bound.

Note, that for $b=2$, Construction F coincides with Construction A. This time, the generalization is not so simple. Instead of the two bits of Construction A which determine the cluster pattern, we have the top layer of $b-1$ bits, and the middle layer of $m+b-1$ bits. The cluster-locating process is done using the bottom layer of m bits, and the middle layer of $m+b-1$ bits.

For some parameters, a slight improvement may be obtained by using the following construction. This construction combines b-burst-correcting codes and codes which can correct rank errors [21]. A μ-$[n\times n,k]$ binary array code C is a k-dimensional linear space of the $n\times n$ matrices such that every nonzero matrix in C has rank of at least μ. Such a code is capable of correcting rank $(\mu-1)/2$ errors. The following construction was given by Roth [23].

Construction G: Let H be an $r\times n$ binary parity-check matrix of a one-dimensional b-burst-correcting code C_1 and let C_2 be a 3-$[r\times r,r(r-2)]$ binary array code capable of correcting rank 1 errors. Construct the following code:

$$C_b^{(S)}=\left\{c\mid HcH^T\in C_2\right\}.$$

Theorem 7: The code $C_b^{(S)}$ has redundancy at most $2r$ and can correct any $b\times b$ rectangular cluster error of rank 1.

Proof: First note that $C_b^{(S)}$ is a linear code whose redundancy is at most $2r$. Let e be a nonzero $n\times n$ binary array, whose nonzero entries are confined to a $b\times b$ rectangular cluster of rank 1. It is easy to see that HeH^T is an array with rank 1. Hence, if $c\in C_b^{(S)}$ was transmitted and $c+e$ was received, then since C_2 can correct rank 1 errors, it follows that we can recover HeH^T.

Now, He is a nonzero $r \times n$ array whose nonzero entries are confined to b consecutive columns. Hence, each of the rows of He contains a burst of length at most b, and since C_1 is a b-burst-correcting code, it follows that we can recover He from HeH^T. Using C_1 again, we can recover e from He. □

Corollary 5: $C_b^{(S)}$ is a b-straight-cluster-correcting code.

Corollary 6: Let C_1 be a perfect 2-burst-correcting code of length 2^m and redundancy $m+1$, $m \geq 4$ (see [9]) and let C_2 be a 3-$[(m+1)\times(m+1),(m+1)(m-1)]$ binary array code (see [21]). Using Construction G with C_1 and C_2 we obtain an optimal $2^m \times 2^m$ 2-cluster-correcting code.

Corollary 7: Let C_1 be a b-burst-correcting code of length 2^m-1, and redundancy $m+b-1$ (see [2]) and let C_2 be a 3-$[(m+b-1)\times(m+b-1),(m+b-1)(m+b-3)]$ binary array code. Using Construction G with C_1 and C_2 we obtain a $(2^m-1)\times(2^m-1)$ b-straight-cluster-correcting code with redundancy at most $2m+2b-2$.

B. The $*$ Model

Construction H: Let H be an $(m+b-1)\times(2^m-1)$ parity-check matrix for a one-dimensional cyclic b-burst-correcting code (see [2]), where $m \geq 3b-2$ if m is even, and $m \geq b$ if m is odd. We denote the columns of H by $h_0, h_1, \ldots, h_{2^m-2}$. Let α be a primitive element of GF(2^m). We construct the following $(2^m-1)\times(2^m-1)\times(2m+3b-2)$ parity-check matrix $H_b^{(S)*}(m)$:

$$h_{ij} = \begin{bmatrix} 1 \\ \boldsymbol{e}_{i \,(\mathrm{mod}\, b)} \\ \boldsymbol{e}_{j \,(\mathrm{mod}\, b)} \\ h_{i-j} \\ \alpha^{j-2i} \end{bmatrix}, \qquad \text{for all } \; 0 \leq i, \; j \leq 2^m - 2$$

where the index of h_{i-j} is taken modulo 2^m-1.

Theorem 8: $H_b^{(S)*}(m)$ is a parity-check matrix of a b-straight-cluster-correcting code in the $*$ model.

Proof: Again, we prove that all nonzero syndromes associated with b-straight-cluster errors are distinct. Let $s=(s_0,s_1,s_2,s_3,s_4)^T$, where the lengths of s_0, s_1, s_2, s_3, and s_4, are 1, $b-1$, $b-1$, $m+b-1$, and m, respectively, be a syndrome. Let wt(s_i) denote the weight of s_i. Table I presents the weights of s_1 and s_2, given the direction and the parity of the number of errors (we do not take a single error into this account as it can be viewed as an error in any direction). Clearly, s_0, s_1, and s_2, determine whether the errors occurred in a row, a column, or a diagonal.

If the errors occurred either in a single row or in a single column, we continue along the same lines of the proof of Theorem 6 to show that all syndromes related to distinct positions are distinct. Hence, we consider the case where the errors occurred in a single diagonal. We first note that in the diagonals of $H_b^{(S)*}(m)$ we have a cyclic ordering from

$$H' = [h_0, h_2, h_4, \ldots, h_{2^m-2}, h_1, h_3, \ldots, h_{2^m-3}].$$

TABLE I
ERROR-DIRECTION INDICATORS FOR THE $*$ MODEL

Direction	Parity	wt(s_0)	wt(s_1)	wt(s_2)
Row	Even	0	0	≥ 1
	Odd	1	0 or 1	≥ 2
Column	Even	0	≥ 1	0
	Odd	1	≥ 2	0 or 1
Diagonal	Even	0	≥ 1	≥ 1
	Odd	1	≥ 2	≥ 2

But H' is also cyclic b-burst-correcting code (see the construction in [2]).

We note, that once it is known in which diagonal the errors occurred, the positions of the errors in the diagonal are uniquely determined by s_3, since the appropriate entries of $H_b^{(S)*}(m)$ form a cyclic shift of a parity-check matrix of a cyclic b-burst-correcting code.

To complete the proof, we show that if two cluster errors along two different diagonals have syndromes

$$s=(s_0,s_1,s_2,s_3,s_4)^T \quad \text{and} \quad s'=(s_0,s_1,s_2,s_3,s_5)^T$$

then $s_4 \neq s_5$. Now, assume that the first cluster error occurs along a diagonal at positions

$$(i_1,j_1),(i_1-d_1,j_1+d_1),(i_1-d_2,j_1+d_2),\ldots,(i_1-d_\ell,j_1+d_\ell)$$

where $0<d_1<d_2<\cdots<d_\ell \leq b-1$. Assume also that the second cluster error occurs along a diagonal at positions

$$(i_2,j_2),(i_2-d_1,j_2+d_1),(i_2-d_2,j_2+d_2),\ldots,(i_2-d_\ell,j_2+d_\ell).$$

Since the first $m+3b-2$ positions in the two syndromes are the same, it follows that $i_1-j_1 \equiv i_2-j_2 \pmod{2^m-1}$. We also have that

$$s_4 = \alpha^{j_1-2i_1}(1+\alpha^{3d_1}+\cdots+\alpha^{3d_\ell})$$

and

$$s_5 = \alpha^{j_2-2i_2}(1+\alpha^{3d_1}+\cdots+\alpha^{3d_\ell}).$$

Since either $3d_\ell \leq 3b-3 < m$ when m is even, or α^3 is a primitive element when $m \geq b$ is odd, it follows that $1+\alpha^{3d_1}+\cdots+\alpha^{3d_\ell} \neq 0$. Hence, if we assume that $s_4=s_5$, then

$$\begin{aligned} j_1-2i_1 &\equiv j_2-2i_2 \pmod{2^m-1} \\ i_1-j_1 &\equiv i_2-j_2 \pmod{2^m-1} \end{aligned}$$

which implies that $i_1=i_2$, a contradiction. Thus, $s_4 \neq s_5$. □

The code defined by $H_b^{(S)*}(m)$ has redundancy $2m+3b-2$. By the sphere-packing bound, a lower bound on the redundancy of a two-dimensional code of size $(2^m-1)\times(2^m-1)$ is $2m+b+1$ for $b\geq 3$ and m large enough.

C. The $\circledast$ Model

Construction I: Let $m \geq b \geq 3$, be both odd integers, and let H be an $(m+b-1)\times(2^m-1)$ parity-check matrix for a one-dimensional cyclic b-burst-correcting code (see [2]). We denote the columns of H as $h_0, h_1, \ldots, h_{2^m-2}$. Let α be a primitive

TABLE II
ERROR-DIRECTION INDICATORS FOR THE $*$ MODEL

Direction	Parity	wt(s_0)	wt(s_1)	wt(s_2)	wt(s_3)
Row	Even	0	0	≥ 1	≥ 1
	Odd	1	0 or 1	≥ 2	≥ 2
Column	Even	0	≥ 1	0	≥ 1
	Odd	1	≥ 2	0 or 1	≥ 2
/ Diagonal	Even	0	≥ 1	≥ 1	0
	Odd	1	≥ 2	≥ 2	0 or 1
\ Diagonal	Even	0	≥ 1	≥ 1	≥ 1
	Odd	1	≥ 2	≥ 2	≥ 2

element of GF(2^m). We construct the following $(2^m-1)\times(2^m-1)\times(2m+4b-3)$ parity-check matrix $H_b^{(S)*}(m)$:

$$h_{ij} = \begin{bmatrix} 1 \\ \boldsymbol{e}_{i\ (\mathrm{mod}\,b)} \\ \boldsymbol{e}_{j\ (\mathrm{mod}\,b)} \\ \boldsymbol{e}_{i+j\ (\mathrm{mod}\,b)} \\ h_{i-2j} \\ \alpha^{i+2j} \end{bmatrix}, \qquad \text{for all } 0 \leq i,j \leq 2^m-2,$$

where the index of h_{i-2j} is taken modulo 2^m-1.

Theorem 9: $H_b^{(S)*}(m)$ is a parity-check matrix of a b-straight-cluster-correcting code in the $*$ model.

Proof: Like before, we prove that all nonzero syndromes associated with a b-straight-cluster errors are distinct. Let $s=(s_0,s_1,s_2,s_3,s_4,s_5)^T$, where the lengths of s_0, s_1, s_2, s_3, s_4, and s_5, are $1, b-1, b-1, b-1, m+b-1$, and m, respectively, be a syndrome. We use s_0, s_1, s_2, and s_3, to determine whether the errors occurred in a row, a column, / diagonal, or \ diagonal exactly as in the proof of Theorem 8 (see Table II).

We note, that once it is known in which direction (row, column, or one of the two diagonal directions) the errors occurred, the proof continues exactly as in the proof of Theorem 8. □

The requirement that m is odd is due to the fact that in one of the diagonal directions, the ordering of the columns of H is $h_0, h_3, h_6, \ldots$. This ordering forms another cyclic b-burst-correcting code if and only if 3 does not divide 2^m-1, i.e., m is odd. The definition $\boldsymbol{e}_{i+j\ (\mathrm{mod}\ b)}$ implies that b should be odd. If b is even then along \ diagonals, we see only half of the set $\{\boldsymbol{e}_0,\ldots,\boldsymbol{e}_{b-1}\}$ and Table II is no longer correct. This can be fixed by taking $\boldsymbol{e}_{i+j\ (\mathrm{mod}\ b+1)}$ to be b-bits-long vectors, so an extra bit enables us to use the construction also for even b.

The code defined by $H_b^{(S)*}(m)$ has redundancy $2m+4b-3$. By the sphere-packing bound, a lower bound on the redundancy of a two-dimensional code of size $(2^m-1)\times(2^m-1)$ is $2m+b+1$ for m large enough.

VI. Lower Bounds on the Redundancy

In this section, we prove some lower bounds on the redundancy of linear b-cluster-correcting codes. These bounds generalize the Reiger bound [20] on the redundancy of b-burst-correcting codes. We handle only the case of the $+$ model. For the rest of this section, C is an $[n_1\times n_2, n_1n_2-r]$ b-cluster-correcting code with codewords of size $n_1\times n_2$, dimension n_1n_2-r, and redundancy r. We call such a code, an $[n_1\times n_2, n_1n_2-r; b]$ code.

By applying the general Singleton-type bound proposed in [7], we obtain the following result.

Theorem 10: If C is an $[n_1\times n_2, n_1n_2-r; b]$ code, then $r\geq 2b$.

This is exactly the bound given by Reiger [20] for the one-dimensional case. Since the proof of the previous theorem does not use the two-dimensional nature of the code, it may obviously be improved. We start with a simple improvement. The proof of this improvement has the same flavor as the proof of Theorem 12, which is a more significant improvement.

Theorem 11: If C is an $[n_1\times n_2, n_1n_2-r; b]$ code with $b\geq 3$, $n_1\geq 3$, and $n_2\geq b$, then $r\geq 2b+1$.

Proof: Assume the contrary, i.e., that $r=2b$. Let H be an $n_1\times n_2\times r$ parity-check matrix for C. We examine the following set of positions:

$$B=\{(x,y)\mid 0\leq x\leq 1 \text{ and } 0\leq y\leq b-1\}\cup\{(2,0)\}.$$

Since $|B|=2b+1$ and $\mathrm{rank}H=2b$, the following equation has a nontrivial solution:

$$\sum_{(i,j)\in B} c_{i,j}h_{i,j}=0$$

where $c_{i,j}\in\{0,1\}$, and $h_{i,j}$ is the column vector of H at position (i,j). We define c_B to be the $n_1\times n_2$ binary word for which position (i,j) is $c_{i,j}$ if $(i,j)\in B$, and zero otherwise. Clearly, c_B is a codeword of C.

If $c_{2,0}=0$ then there exists a nontrivial linear combination of $2b$ columns of H which is equal to zero. Furthermore, this combination is the sum of two clusters of size b

$$\{(0,y)\mid 0\leq y\leq b-1\} \text{ and } \{(1,y)\mid 0\leq y\leq b-1\}.$$

But, this is a contradiction to the fact that C is a b-cluster-correcting code. Therefore, $c_{2,0}=1$.

Similarly, if $c_{0,y}=0$ for some $0\leq y\leq b-1$ then we have a nontrivial linear combination of $2b$ columns of H which is equal to zero. It can be easily verified that we can partition the corresponding $2b$ positions into two clusters of size b. This is demonstrated in Fig. 6. But, this is a contradiction to the fact that C is a b-cluster-correcting code. Therefore, $c_{0,y}=1$ for all $0\leq y\leq b-1$.

Similarly, we define the set

$$B'=\{(x,y)\mid 0\leq x\leq 1 \text{ and } 0\leq y\leq b-1\}\cup\{(2,b-1)\}.$$

Again, there is a nontrivial solution to

$$\sum_{(i,j)\in B'} c'_{i,j}h_{i,j}=0$$

for which $c'_{0,y}=1$ for all $0\leq y\leq b-1$, and $c'_{2,b-1}=1$. Hence, $c'_{B'}$ defined similarly to c_B, is also a codeword of C.

Since C is linear, $c_B+c'_{B'}$ is also a codeword. However, $c_B+c'_{B'}$ has weight at most $b+2$, for which the corresponding $2b$ positions can be partitioned into two clusters of size less or equal to b (see Fig. 7). This contradicts the fact that C is a b-cluster-correcting code. Thus, $r\geq 2b+1$. □

The result of Theorem 11 can be improved by using similar arguments.

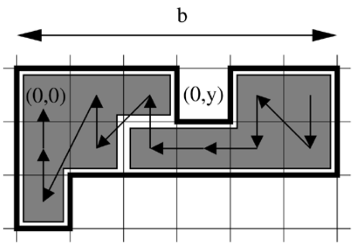

Fig. 6. A partition of $2b$ positions into two clusters of size b.

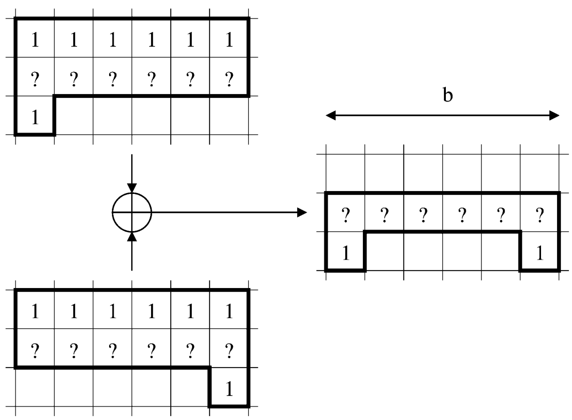

Fig. 7. Sum of two clusters of size $2b+1$ each, gives a cluster pattern of size $b+2$.

Theorem 12: If C is an $[n_1 \times n_2, n_1n_2 - r; b]$ code with $b \geq 5$, $n_1 \geq 4$, $n_2 \geq 2b - 2\left\lfloor \frac{2b}{3} \right\rfloor + 2$, and $n_1n_2 \geq 2b + 2t$, where $t = \lfloor \frac{b-2}{3} \rfloor$, then $r \geq 2b + t$.

Proof: Let $a = \left\lfloor \frac{2b}{3} \right\rfloor$, and assume to the contrary, that $r \leq 2b + t - 1$. Let H be an $n_1 \times n_2 \times r$ parity-check matrix of the code C. We examine the following set of $2b$ positions:

$$B = \{(x, y) \mid x = 0, 2, \text{ and } 1 \leq y \leq a\} \cup \{(1, y) \mid 1 \leq y \leq 2b - 2a\}.$$

Let B_1, B_2, be two sets of positions which satisfy

- B_i, $i = 1, 2$, is a cluster of size t;
- $B_1 \cap B_2 = B_1 \cap B = B_2 \cap B = \emptyset$;
- $(1, 0) \in B_1$ and $(1, 2b - 2a + 1) \in B_2$.

Note, that it is possible to choose two such sets given the values of t and a. Denote $B' = B \cup B_1$ and $B'' = B \cup B_2$. Since $|B'| = 2b + t$ and $r < 2b + t$, the following equation has a nontrivial solution:

$$\sum_{(i,j)\in B'} c_{i,j} h_{i,j} = 0$$

where $c_{i,j} \in \{0, 1\}$, and $h_{i,j}$ is the column vector of H at position (i, j). Hence, $c_{B'}$ is a codeword.

If $c_{B'}$ has more than t zeros in rows 0 and 2 in the positions of B, then $c_{B'}$ has weight at most $2b - 1$ and it can be shown that the corresponding positions can be partitioned into two clusters of size less or equal to b. Thus, the number of zeros in $c_{B'}$, in the positions of B in rows 0 and 2, is at most t. Similarly, there is a codeword $c_{B''}$. The number of zeros in $c_{B''}$ in rows 0 and 2 in the position of B is at most t.

We contend that $c_{B'} \neq c_{B''}$. Otherwise, all the nonzero positions in both codewords must be confined to the positions of B. But $|B| = 2b$ and it can obviously be partitioned into two clusters of size b, which is a contradiction.

Since C is linear, it follows that $c = c_{B'} + c_{B''} \neq 0$ is also a codeword. The weight of c, in the positions of B in rows 0 and 2, is at most $2t$. Hence, the weight of c is at most $4t + 2b - 2a$. Clearly $4t + 2b - 2a \leq 2b - 1$, and it is easy to verify that the corresponding positions can be partitioned into two clusters of size less or equal to b. This contradicts the fact that C is a b-cluster-correcting code. Thus, $r \geq 2b + t$. □

We note that the bound of Theorem 12 also holds when $n_1 = 3$ as long as we can shift B to the right so that B_1 and B_2 have enough space.

Theorem 11 can be modified to handle clusters of size 2 and arrays with width 2.

Theorem 13: If C is an $[n_1 \times n_2, n_1n_2 - r; b]$ code with $b \geq 2$, $n_1 \geq 2$, and $n_2 \geq 2b$, then $r \geq 2b + 1$.

Proof: The proof is similar to that of Theorem 11, where

$$B = \{(0, y) \mid 0 \leq y \leq 2b - 1\} \cup \{(1, 1)\}$$

and

$$B' = \{(0, y) \mid 0 \leq y \leq 2b - 1\} \cup \{(1, 2)\}$$

and for

$$\sum_{(i,j)\in B} c_{i,j} h_{i,j} = 0$$

and

$$\sum_{(i,j)\in B'} c'_{i,j} h_{i,j} = 0$$

it is shown that $c_{0,0} = 1$, $c_{0,2b-1} = 1$, $c'_{0,0} = 1$, and $c'_{0,2b-1} = 1$. □

Similarly to Theorem 12, we have the following result.

Theorem 14: If C is a $[2 \times n_2, 2n_2 - r; b]$ code with $b \geq 6$, and $n_2 \geq b + 1 + t$, where $t = \left\lfloor \frac{b-2}{4} \right\rfloor$, then $r \geq 2b + t$.

Proof: Let $t = \left\lfloor \frac{b-2}{4} \right\rfloor$, and assume to the contrary, that $r \leq 2b + t - 1$. Let H be a $2 \times n_2 \times r$ parity-check matrix of the code C. We examine the following set of $2b$ positions:

$$B = \{(x, y) \mid 0 \leq x \leq 1 \text{ and } 0 \leq y \leq b - 2 + 2x\}.$$

We also define

$$B_1 = \{(0, y) \mid b - 1 \leq y \leq b + t - 2\}$$
$$B_2 = \{(1, y) \mid b + 1 \leq y \leq b + t\}.$$

Denote $B' = B \cup B_1$ and $B'' = B \cup B_2$. Since $|B'| = 2b + t$ and $r < 2b + t$, the following equation has a nontrivial solution:

$$\sum_{(i,j)\in B'} c_{i,j} h_{i,j} = 0$$

where $c_{i,j} \in \{0, 1\}$, and $h_{i,j}$ is the column vector of H at position (i, j). Hence, $c_{B'}$ is a codeword.

If $c_{B'}$ has more than t zeros in row 0 in the positions of B, then $c_{B'}$ has weight at most $2b - 1$ and it can be shown that the corresponding positions can be partitioned into two clusters of size less or equal to b. Thus, the number of zeros in $c_{B'}$, in the positions of B in row 0, is at most t. Similarly, there is a codeword $c_{B''}$. The number of zeros in $c_{B''}$ in row 0 in the positions of B is at most t.

Just like in the proof of Theorem 12, $c_{B'} \neq c_{B''}$. Since C is linear, it follows that $c = c_{B'} + c_{B''} \neq 0$ is also a codeword. The weight of c, in the positions of B in row 0, is at most $2t$. Hence, the weight of c is at most $4t+b+1$. Clearly, $4t+b+1 \leq 2b-1$, and it is easy to verify that the corresponding positions can be partitioned into two clusters of size less of equal to b. This contradicts the fact that C is a b-cluster-correcting code. Thus, $r \geq 2b+t$. □

The bounds in this section can be shown to be optimal for small values of b. However, we believe that in the general case there should be a much stronger bound.

VII. Conclusion and Open Problems

We considered codes which correct cluster errors of arbitrary shape in two dimensions. Some of the codes we constructed were proved to be optimal and some are very close to the sphere-packing bound. We considered small cluster errors, and also errors which occur only in one line of the array. When the error size is two, we also considered a generalization to multidimensional arrays. Codes were designed for three different connectivity models: two grid models, with four and eight neighbors, and the $*$ model. Finally, we improved the well-known Reiger lower bound on the redundancy of linear codes which correct cluster errors of size b.

Since all the constructions presented are linear, they admit a simple method of encoding by way of a generating matrix. More importantly, all the codes allow for a relatively simple decoding procedure. Since the codes are built using a cluster-identification layer and a cluster-locating layer in the resulting syndrome, the following decoding procedure is possible.

1) The syndrome of a received word is calculated.
2) The cluster-identification layer of the syndrome is examined and compared against a table of possible results.
3) A set of equations using the cluster-locating layer should be solved to complete the procedure.

There are two things to note, however. First, for b-straight-cluster-correcting codes, the second stage of identifying the cluster pattern requires a decoder for the one-dimensional cyclic b-burst-correcting code. The second is that for the small sizes of clusters considered in this paper, the table required in the second stage is small in size. However, the number of distinct cluster patterns grows exponentially in the cluster size and may require a different solution for large clusters. It should be noted that the exact number of distinct cluster patterns in the different connectivity models is unknown [25] and is an interesting combinatorial question in itself.

Though all the constructions described produce codes of size $(2^m-1)\times(2^m-1)$, a simple shortening, by removing contiguous rows and/or columns from the parity-check matrix, gives us codes of any size. Furthermore, for the optimal (nearly optimal) constructions, as long as the resulting $n_1 \times n_2$ code after shortening fulfills $2^{m-1}-1 \leq n_1, n_2 \leq 2^m-1$, we are assured that the redundancy of the code is no more than two redundancy bits over being optimal (nearly optimal).

The constructions in this paper appear to be the first cluster-correcting codes built using a direct algebraic approach. Though the constructions for arbitrary-shaped clusters handle only small clusters in an *ad hoc* fashion, we hope that the methods employed may be further generalized to larger clusters.

For future research we would like to see constructions for cluster-correcting codes for cluster errors whose size is greater than three. We would like to know whether our constructions for 3-cluster-correcting codes and b-straight-cluster-correcting codes are optimal either by improving the sphere-packing lower bound, or by improving our constructions. Finally, we would like to see further improvement of the lower bound on the redundancy compared to the cluster size b in linear b-cluster-correcting codes.

Acknowledgment

The authors wish to thank Ron M. Roth for providing Construction G and for many helpful discussions and Eitan Yaakobi for pointing out errors in an earlier version of the paper. The authors also wish to thank the anonymous reviewers, whose comments improved the presentation of the paper.

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