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Optimal tristance anticodes in certain graphs

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Abstract

For $z_1, z_2, z_3 \in \mathbb{Z}^n$, the *tristance* $d_3(z_1, z_2, z_3)$ is a generalization of the L_1-distance on $\mathbb{Z}^n$ to a quantity that reflects the relative dispersion of three points rather than two. A *tristance anticode* $\mathcal{A}_d$ of diameter d is a subset of $\mathbb{Z}^n$ with the property that $d_3(z_1, z_2, z_3) \leqslant d$ for all $z_1, z_2, z_3 \in \mathcal{A}_d$. An anticode *is optimal* if it has the largest possible cardinality for its diameter d. We determine the cardinality and completely classify the optimal tristance anticodes in $\mathbb{Z}^2$ for all diameters $d \geqslant 1$. We then generalize this result to two related distance models: a different distance structure on $\mathbb{Z}^2$ where $d(z_1, z_2) = 1$ if z_1, z_2 are adjacent either horizontally, vertically, or *diagonally*, and the distance structure obtained when $\mathbb{Z}^2$ is replaced by the hexagonal lattice A_2. We also investigate optimal tristance anticodes in $\mathbb{Z}^3$ and optimal quadristance anticodes in $\mathbb{Z}^2$, and provide bounds on their cardinality. We conclude with a brief discussion of the applications of our results to multi-dimensional interleaving schemes and to connectivity loci in the game of Go.

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1. Introduction

Given two points $z = (z_1, z_2, \ldots, z_n)$ and $z' = (z'_1, z'_1, \ldots, z'_n)$ in $\mathbb{Z}^n$, the L_1-distance between z and z' is defined as $d(z, z') = |z_1 - z'_1| + |z_2 - z'_2| + \cdots + |z_n - z'_n|$. Alternatively, let $\mathcal{G}_n^{\boxplus} = (V, E)$ denote the *grid graph* of $\mathbb{Z}^n$ whose vertex set is $V = \mathbb{Z}^n$ and whose edges are defined as follows: $\{z, z'\} \in E$ if and only if $d(z, z') = 1$. Then the L_1-distance between z and z' in $\mathbb{Z}^n$ is the number of edges in the shortest path joining z and z' in $\mathcal{G}_n^{\boxplus}$. The latter point of view leads to a natural generalization of the L_1-distance on $\mathbb{Z}^n$ to a quantity that reflects the relative dispersion of three points rather than two.

Definition 1. Let $z_1, z_2, z_3 \in \mathbb{Z}^n$. Then the *tristance* $d_3(z_1, z_2, z_3)$ is defined as the number of edges in a minimal spanning tree for z_1, z_2, z_3 in the grid graph $\mathcal{G}_n^{\boxplus}$ of $\mathbb{Z}^n$.

The notion of tristance defined above can be further generalized in two different ways. First, the *quadristance* $d_4(z_1, z_2, z_3, z_4)$, the *quintistance* $d_5(z_1, z_2, z_3, z_4, z_5)$, and more generally the *r-dispersion* $d_r(z_1, z_2, \ldots, z_r)$ may be defined [9,14,18] as the number of edges in a minimal spanning tree for $z_1, z_2, \ldots, z_r$ in the grid graph $\mathcal{G}_n^{\boxplus}$. Herein, we consider only the tristance $d_3(z_1, z_2, z_3)$ and, briefly, the quadristance $d_4(z_1, z_2, z_3, z_4)$ in §4.2.

Another way to generalize Definition 1 is to replace the grid graph $\mathcal{G}_n^{\boxplus}$ by a different graph. We will consider two alternative graphs $\mathcal{G}_2^{\infty}$ and $\mathcal{G}_2^{\hexagon}$ that are useful in applications to two-dimensional interleaving [5,6,9,18]. The graph $\mathcal{G}_2^{\infty}$ has $\mathbb{Z}^2$ as its vertex set, with $z = (x, y)$ and $z' = (x', y')$ in $\mathbb{Z}^2$ being adjacent if and only if

$$d^{\infty}(z, z') \stackrel{\text{def}}{=} \max\{|x - x'|, |y - y'|\} = 1.$$

Thus tristance in $\mathcal{G}_2^{\infty}$ may be thought of as a generalization of the L_∞-distance on $\mathbb{Z}^2$. The vertex set of the graph $\mathcal{G}_2^{\hexagon}$ is the hexagonal lattice $A_2 = \{(1/2\, v, u + \sqrt{3}/2\, v) : u, v \in \mathbb{Z}\}$, with two points $z = (x, y)$ and $z' = (x', y')$ in A_2 being adjacent iff

$$d_{\mathrm{E}}(z, z') \stackrel{\text{def}}{=} \sqrt{(x - x')^2 + (y - y')^2} = 1.$$

The three graphs $\mathcal{G}_2^{\boxplus}$, $\mathcal{G}_2^{\infty}$, and $\mathcal{G}_2^{\hexagon}$ are illustrated in Fig. 1. We will sometimes refer to the graphs $\mathcal{G}_2^{\boxplus}$, $\mathcal{G}_2^{\infty}$, and $\mathcal{G}_2^{\hexagon}$ as the grid graph, the *infinity graph*, and the *hexagonal graph*, respectively (or the $\boxplus$ model, the ∞ model, and the $\hexagon$ model, for short).

Given a set $\mathcal{S}$ and a definition of distance between points of $\mathcal{S}$, a *code* $\mathcal{C} \subseteq \mathcal{S}$ of minimum distance d is characterized by the property that the distance between any two distinct points of $\mathcal{C}$ is *at least* d. Similarly, an *anticode* $\mathcal{A} \subseteq \mathcal{S}$ of diameter d is characterized by the property that the distance between any two distinct points of $\mathcal{A}$ is *at most* d. One is usually interested in codes and anticodes of the largest possible cardinality for a given minimum distance or diameter—such codes/anticodes are said to be *optimal*. An encyclopedic survey of optimal codes in the Hamming graph may be found in [12,16]; for codes in other graphs, see [3,4,8,11,17,18]. Optimal anticodes in the Hamming metric and related distance-regular graphs have been studied in [1,2,8,13,17] and other papers.

The concepts of a code and an anticode can be generalized using the notion of tristance in Definition 1. Thus a tristance code $\mathcal{C} \subseteq \mathbb{Z}^n$ of minimum tristance d is a subset of $\mathbb{Z}^n$ such

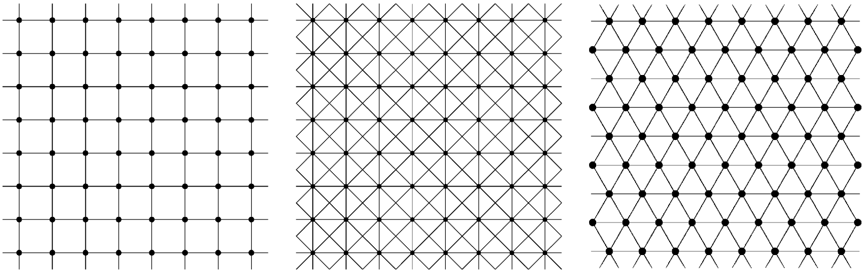

Fig. 1. The grid graph $\mathcal{G}_2^{\boxplus}$, the infinity graph $\mathcal{G}_2^{\infty}$, and the exagonal graph $\mathcal{G}_2^{\hexagon}$.

that $d_3(z_1, z_2, z_3) \geqslant d$ for all $z_1, z_2, z_3 \in \mathcal{C}$. Numerous results on optimal tristance codes in $\mathbb{Z}^2$ can be found in [9,14,18]. Optimal tristance *anticodes* are the subject of this paper.

Definition 2. A set $\mathcal{A}_d \subset \mathbb{Z}^n$ is a *tristance anticode* of diameter d if $d_3(z_1, z_2, z_3) \leqslant d$ for all $z_1, z_2, z_3 \in \mathcal{A}_d$; it is *optimal* if it has the largest possible cardinality for its diameter d.

Observe that Definition 2 can be extended in the obvious way to other graphical models (such as the ∞ and the $\hexagon$ models) as well as to higher dispersions (such as quadristance).

One can also define tristance anticodes *centered* about a given point or a pair of points. Given $z_0 \in \mathbb{Z}^n$, a set $\mathcal{A}_d(z_0) \subset \mathbb{Z}^n$ is said to be a tristance anticode of diameter *d centered about* z_0 if $d_3(z_0, z_1, z_2) \leqslant d$ for all $z_1, z_2 \in \mathcal{A}_d(z_0)$. Given distinct $z_1, z_2 \in \mathbb{Z}^n$, a set $\mathcal{A}_d(z_1, z_2) \subset \mathbb{Z}^n$ is said to be a tristance anticode of diameter *d centered about* z_1 *and* z_2 if $d_3(z_1, z_2, z) \leqslant d$ for all $z \in \mathcal{A}_d(z_1, z_2)$. Once again, we are interested in *optimal* centered tristance anticodes that have the largest possible cardinality for their diameter and center(s). We note that the corresponding problem for conventional (distance) anticodes is trivial. The unique optimal anticode of diameter d centered about a given point $z_0 \in \mathbb{Z}^n$ is $\mathscr{S}_d(z_0) = \{z \in \mathbb{Z}^n : d(z, z_0) \leqslant d\}$, which is just a *sphere of radius d about* z_0.

The rest of this paper is organized as follows. The next section is concerned with optimal tristance anticodes in $\mathbb{Z}^2$. We determine the cardinality and classify the optimal tristance anticodes $\mathcal{A}_d(z_1, z_2)$, $\mathcal{A}_d(z_0)$, and $\mathcal{A}_d$ in the grid graph $\mathcal{G}_2^{\boxplus}$, for all $d \geqslant 1$. We also introduce in §2 certain methods and techniques that will be useful throughout this paper. In §3, we extend the results of §2 to the ∞ model and the $\hexagon$ model. In §4, we pursue generalizations to higher dimensions and to higher dispersions: we investigate optimal tristance anticodes in $\mathbb{Z}^3$ and optimal quadristance anticodes in $\mathbb{Z}^2$. In §5, we discuss some of the applications of our results to multi-dimensional interleaving schemes with repetitions [6,5,9,18] to multicasting in processor networks, and to the study of connectivity loci in the game of Go.

2. Optimal tristance anticodes in the grid graph

We will first need some auxiliary results. Trivially, the L_1-distance between two points (x_1, y_1) and (x_2, y_2) in $\mathbb{Z}^2$ can be written as $(\max_{1 \leqslant i \leqslant 2} x_i - \min_{1 \leqslant i \leqslant 2} x_i) + (\max_{1 \leqslant i \leqslant 2} y_i - \min_{1 \leqslant i \leqslant 2} y_i)$. The following theorem of [9] shows that a similar expression holds for tristance in $\mathbb{Z}^2$.

Theorem 1. *Let $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2)$, $z_3 = (x_3, y_3)$ be distinct points in $\mathbb{Z}^2$. Then*

$$d_3(z_1, z_2, z_3) = \left(\max_{1\leqslant i\leqslant 3} x_i - \min_{1\leqslant i\leqslant 3} x_i\right) + \left(\max_{1\leqslant i\leqslant 3} y_i - \min_{1\leqslant i\leqslant 3} y_i\right). \tag{1}$$

Next, we recall some results on optimal *distance* anticodes in the grid graph $\mathcal{G}_2^{\boxplus}$. Let d be even, let $z_0 = (x_0, y_0)$ be an arbitrary point in $\mathbb{Z}^2$, and consider the set

$$\mathscr{S}_{d/2}(z_0) = \left\{(x, y) \in \mathbb{Z}^2 \;:\; |x - x_0| + |y - y_0| \leqslant \frac{d}{2}\right\} \tag{2}$$

which is the L_1-sphere of radius $d/2$ about z_0. By triangle inequality, for all $z_1, z_2 \in \mathscr{S}_{d/2}(z_0)$ we have $d(z_1, z_2) \leqslant d(z_1, z_0) + d(z_0, z_2) \leqslant d$, so $\mathscr{S}_{d/2}(z_0)$ is an anticode of diameter d. It is easy to see that $|\mathscr{S}_{d/2}(z_0)| = d^2/2 + d + 1$. On the other hand, it is shown in [6] that $\mathbb{Z}^2$ can be partitioned into $d^2/2 + d + 1$ codes such that the minimum L_1-distance of each code is $d + 1$. Obviously, any anticode of diameter d can contain at most one point from each such code. It follows that the anticode $\mathscr{S}_{d/2}(z_0)$ is optimal for all even d. For odd d, we let z_0 be an arbitrary point in $(1/2, 0) + \mathbb{Z}^2$ or in $(0, 1/2) + \mathbb{Z}^2$. Then $\mathscr{S}_{d/2}(z_0)$ defined in (2) is again an anticode of diameter d (by triangle inequality), and $|\mathscr{S}_{d/2}(z_0)| = d^2/2 + d + 1/2$. Once again, it is shown in [6] that $\mathbb{Z}^2$ can be partitioned into $d^2/2 + d + 1/2$ codes with even minimum distance $d + 1$, so the anticode $\mathscr{S}_{d/2}(z_0)$ is optimal for all odd d.

2.1. Uniqueness of optimal anticodes in the grid graph

We now show that optimal distance anticodes in $\mathcal{G}_2^{\boxplus}$ are unique: if $\mathcal{A}$ is an optimal anticode of diameter d, then $\mathcal{A} = \mathscr{S}_{d/2}(z_0)$, where $z_0 \in \mathbb{Z}^2$ if d is even, whereas if d is odd then $z_0 \in (1/2, 0) + \mathbb{Z}^2$ or $z_0 \in (0, 1/2) + \mathbb{Z}^2$. This result is established in a series of lemmas.

A set $\mathcal{S} \subseteq \mathbb{Z}^2$ is *vertically contiguous* if it has the following property: if $(x, y_1) \in \mathcal{S}$ and $(x, y_2) \in \mathcal{S}$, then $(x, y) \in \mathcal{S}$ for all y in the range $\min\{y_1, y_2\} \leqslant y \leqslant \max\{y_1, y_2\}$. Similarly, $\mathcal{S} \subseteq \mathbb{Z}^2$ is *horizontally contiguous* if the fact that $(x_1, y) \in \mathcal{S}$ and $(x_2, y) \in \mathcal{S}$ implies that $(x, y) \in \mathcal{S}$ for all $\min\{x_1, x_2\} \leqslant x \leqslant \max\{x_1, x_2\}$.

Lemma 2. *Let $\mathcal{A}$ be an optimal anticode of diameter d in the grid graph $\mathcal{G}_2^{\boxplus}$. Then $\mathcal{A}$ is both vertically contiguous and horizontally contiguous.*

Proof. Suppose that $z_1 = (x_0, y_1)$ and $z_2 = (x_0, y_2)$ are points in $\mathcal{A}$. Assume w.l.o.g. that $y_2 > y_1$, and consider a point $z_3 = (x_0, y_3)$ with $y_1 \leqslant y_3 \leqslant y_2$. If $z = (x, y) \in \mathcal{A}$, then

$$\begin{aligned} d(z, z_3) &= |x - x_0| + |y - y_3| \leqslant |x - x_0| + \max\{|y - y_1|, |y - y_2|\} \\ &= \max\{d(z, z_1), d(z, z_2)\} \end{aligned}$$

Thus $d(z, z_3) \leqslant d$ for all $z \in \mathcal{A}$, and if $\mathcal{A}$ is optimal, it must contain the point $z_3 = (x_0, y_3)$. Hence $\mathcal{A}$ is vertically contiguous. By a similar argument, $\mathcal{A}$ is horizontally contiguous. □

Given $z = (x, y) \in \mathbb{Z}^2$, we say that the points $(x-1, y)$ and $(x+1, y)$ are the *horizontal neighbors* of z, while the points $(x, y-1)$ and $(x, y+1)$ are the *vertical neighbors* of z.

Lemma 3. *Let $\mathcal{A}$ be an optimal anticode of diameter d in the grid graph $\mathcal{G}_2^{\boxplus}$. If $\mathcal{A}$ contains the two horizontal neighbors of a point $z \in \mathbb{Z}^2$, or if $\mathcal{A}$ contains the two vertical neighbors of z, then $\mathcal{A}$ necessarily contains z itself and* all the four *neighbors of z.*

Proof. Suppose $\mathcal{A}$ contains the points $z_1 = (x_0 - 1, y_0)$ and $z_2 = (x_0 + 1, y_0)$. Since $\mathcal{A}$ is horizontally contiguous by Lemma 2, it also contains the point $z_0 = (x_0, y_0)$. Moreover, if $z = (x, y)$ is any point in $\mathcal{A}$, then $d(z, z_0) = \max\{d(z, z_1), d(z, z_2)\} - 1 \leqslant d - 1$. Now, let $z_3 = (x_0, y_0 + 1)$. Then we have

$$d(z, z_3) = |x - x_0| + |y - y_0 - 1| \leqslant |x - x_0| + |y - y_0| + 1 = d(z, z_0) + 1 \leqslant d.$$

Hence, if $\mathcal{A}$ is optimal, it must contain $z_3 = (x_0, y_0 + 1)$. By a similar argument, $\mathcal{A}$ also contains the point $(x_0, y_0 - 1)$. The claim for vertical neighbors follows by symmetry. □

Given a set $\mathcal{S} \subseteq \mathbb{Z}^2$, let $\mathcal{G}_2^{\boxplus}(\mathcal{S})$ denote the induced subgraph of $\mathcal{G}_2^{\boxplus}$, consisting of $\mathcal{S}$ and the edges of $\mathcal{G}_2^{\boxplus}$ with both endpoints in $\mathcal{S}$. We say that $z \in \mathcal{S}$ is an *internal point* of $\mathcal{S}$ if z has degree 4 in $\mathcal{G}_2^{\boxplus}(\mathcal{S})$; otherwise we say that z is a *boundary point* of $\mathcal{S}$.

Lemma 4. *An optimal anticode of diameter d in $\mathcal{G}_2^{\boxplus}$ has at most $2d$ boundary points.*

Proof. Let $\mathcal{A}$ be an optimal anticode of diameter d in $\mathcal{G}_2^{\boxplus}$, and define the integers $x_{\min}$, $x_{\max}$ as follows: $x_{\min} = \min\{x : (x, y) \in \mathcal{A}\}$ and $x_{\max} = \max\{x : (x, y) \in \mathcal{A}\}$. Clearly $\Delta = x_{\max} - x_{\min} \leqslant d$. Let us partition $\mathcal{A}$ into $\Delta + 1$ vertical segments

$$\mathcal{V}_i \stackrel{\text{def}}{=} \Big\{(x, y) \in \mathcal{A} \;:\; x = x_{\min} + i\Big\} \qquad \text{for } i = 0, 1, \ldots, \Delta. \tag{3}$$

Since $\mathcal{A}$ is vertically contiguous by Lemma 2, for each $i = 0, 1, \ldots, \Delta$, the vertical segment $\mathcal{V}_i$ in (3) can be written as

$$\mathcal{V}_i = \Big\{(x_{\min}+i, y_{\min,i}), (x_{\min}+i, y_{\min,i} + 1), \ldots, (x_{\min}+i, y_{\max,i})\Big\} \tag{4}$$

for some integers $y_{\min,i} \leqslant y_{\max,i}$. Notice that for each point $z \in \mathcal{V}_i$, except $(x_{\min}+i, y_{\min,i})$ and $(x_{\min}+i, y_{\max,i})$, both vertical neighbors of z are in $\mathcal{V}_i$, and hence also in $\mathcal{A}$. Lemma 3 thus implies that z has degree 4 in $\mathcal{G}_2^{\boxplus}(\mathcal{A})$. It follows that each $\mathcal{V}_i$ contains at most two boundary points of $\mathcal{A}$, so that $\mathcal{A}$ has at most $2(\Delta + 1)$ boundary points altogether. If $\Delta \leqslant d - 1$ we are done, so it remains to consider the case $\Delta = d$. But then $|\mathcal{V}_0| = |\mathcal{V}_\Delta| = 1$ (if $y_{\min,0} = y_{\max,0} = y_{\min,\Delta} = y_{\max,\Delta}$ does not hold, there are points in $\mathcal{V}_0 \cup \mathcal{V}_\Delta$ at distance $\geqslant d + 1$ from each other). Thus $\mathcal{A}$ has at most $2d$ boundary points in this case as well. □

Lemma 5. *Let $\mathcal{A}$ be an anticode of diameter $d \geqslant 2$ in the grid graph $\mathcal{G}_2^{\boxplus}$. Then the set of internal points of $\mathcal{A}$, if nonempty, forms an anticode of diameter $d - 2$.*

Proof. By convention, a set of size $\leqslant 1$ has diameter zero. Otherwise, if z_1, z_2 are distinct internal points of $\mathcal{A}$, then all of their neighbors are also in $\mathcal{A}$. Observe that the set of 4 neighbors of z_1 always contains at least one point z such that $d(z, z_2) = d(z_1, z_2) + 1$. It follows that among the neighbors of z_1 and z_2, there are (at least) two points at distance $d(z_1, z_2) + 2$ from each other. Hence if $\mathcal{A}$ has diameter d, then $d(z_1, z_2) \leqslant d - 2$. □

Theorem 6. *Let $\mathcal{A}$ be an optimal anticode of diameter d in the grid graph $\mathcal{G}_2^{\boxplus}$. Then $\mathcal{A} = \mathscr{S}_{d/2}(z_0)$, where $z_0 \in \mathbb{Z}^2$ if d is even and $z_0 \in \{(1/2, 0) + \mathbb{Z}^2\} \cup \{(0, 1/2) + \mathbb{Z}^2\}$ otherwise.*

Proof. We will only prove the theorem for even d; the proof for odd d is similar. We proceed by induction on d. For $d = 2$, it can be readily verified that an anticode of diameter 2 and size 5 is necessarily the L_1-sphere $\mathscr{S}_1(z_0)$ for some $z_0 \in \mathbb{Z}^2$. Now, let $\mathcal{A}$ be an anticode of diameter d and cardinality $|\mathcal{A}| = d^2/2 + d + 1$. Let $\mathscr{D}(\mathcal{A})$ denote the set of internal points of $\mathcal{A}$. Then $\mathscr{D}(\mathcal{A})$ is an anticode of diameter $d - 2$ by Lemma 5, and

$$|\mathscr{D}(A)| \geqslant |\mathcal{A}| - 2d = \frac{(d-2)^2}{2} + (d-2) + 1$$

by Lemma 4. It follows that $\mathcal{A}$ has *exactly* $2d$ boundary points and $\mathscr{D}(\mathcal{A})$ is an *optimal* anticode of diameter $d-2$. Hence, by induction hypothesis, $\mathscr{D}(\mathcal{A}) = \mathscr{S}_{(d/2)-1}(z_0)$ for some $z_0 \in \mathbb{Z}^2$. Referring to (3) and (4), we see that each vertical segment $\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_{\Delta-1}$ in $\mathcal{A}$ must have exactly two boundary points of $\mathcal{A}$, and $|\mathcal{V}_0| = |\mathcal{V}_\Delta| = 1$. In other words, there is a unique way to adjoin $2d$ boundary points to $\mathscr{D}(\mathcal{A}) = \mathscr{S}_{(d/2)-1}(z_0)$ to obtain an optimal anticode, and it is easy to see that the result is precisely the L_1-sphere $\mathscr{S}_{d/2}(z_0)$. □

The foregoing theorem, which is the main result of this subsection, establishes the uniqueness of optimal distance anticodes in $\mathcal{G}_2^{\boxplus}$. All such anticodes are L_1-spheres of radius $d/2$. We are now in a position to begin the classification of optimal *tristance* anticodes in $\mathcal{G}_2^{\boxplus}$.

2.2. Centered tristance anticodes in the grid graph

Recall that $\mathcal{A}_d(z_0) \subset \mathbb{Z}^2$ is a tristance anticode of diameter d *centered about* $z_0 \in \mathbb{Z}^2$, if $d_3(z_0, z_1, z_2) \leqslant d$ for all $z_1, z_2 \in \mathcal{A}_d(z_0)$. First assume that d is even, and consider the L_1-sphere $\mathscr{S}_{d/2}(z_0)$. For all $z_1, z_2 \in \mathscr{S}_{d/2}(z_0)$, we have $d_3(z_0, z_1, z_2) \leqslant d(z_0, z_1) + d(z_0, z_2) \leqslant d$. It follows that $\mathscr{S}_{d/2}(z_0)$ is a centered tristance anticode of diameter $d = 2t$ and cardinality $2t^2 + 2t + 1$. Now suppose that $d = 2t + 1$ is odd, and consider the L_1-sphere $\mathscr{S}_{d/2}(z_0 + \xi)$, where $\xi = (1/2, 0)$. Let $z_0 = (x_0, y_0)$, and let $z_1 = (x_1, y_1)$ be any point in $\mathscr{S}_{d/2}(z_0 + \xi)$. Then $d(z_0, z_1) \leqslant t + 1$ and, moreover, $d(z_0, z_1) \leqslant t$ unless $x_1 > x_0$. It follows that $\mathscr{S}_{d/2}(z_0 + \xi)$ is a tristance anticode centered about z_0, of diameter $d = 2t + 1$ and cardinality $2t^2 + 4t + 2$. The following theorem shows that the anticodes constructed above are the *unique* optimal tristance anticodes centered about z_0 in the grid graph $\mathcal{G}_2^{\boxplus}$.

Theorem 7. *Let $\mathcal{A}_d(z_0)$ be an optimal tristance anticode of diameter d in $\mathcal{G}_2^{\boxplus}$ centered about $z_0 \in \mathbb{Z}^2$. If d is even, then $\mathcal{A}_d(z_0) = \mathscr{S}_{d/2}(z_0)$. If d is odd, then $\mathcal{A}_d(z_0) = \mathscr{S}_{d/2}(z_0 + \xi)$ for some $\xi \in \{(1/2, 0), (0, 1/2), (-1/2, 0), (0, -1/2)\}$.*

Proof. We will prove the theorem for even d only; the proof for odd d is similar. For all $z_1, z_2 \in \mathcal{A}_d(z_0)$, we have $d(z_1, z_2) \leqslant d_3(z_0, z_1, z_2) \leqslant d$ by definition. Hence $\mathcal{A}_d(z_0)$ is also a distance anticode of diameter d and

$$|\mathcal{A}_d(z_0)| \leqslant d^2/2 + d + 1. \tag{5}$$

We have shown that $\mathscr{S}_{d/2}(z_0)$ is a tristance anticode of diameter d centered about z_0. Since $|\mathscr{S}_{d/2}(z_0)| = d^2/2 + d + 1$, this anticode is optimal in view of (5). Moreover, by Theorem 6, equality in (5) is possible only if $\mathcal{A}_d(z_0) = \mathscr{S}_{d/2}(z_0)$. □

Next, we consider the anticodes $\mathcal{A}_d(z_1, z_2) \subset \mathbb{Z}^2$ centered about a pair of points z_1 and z_2 and defined by the property that $d_3(z_1, z_2, z) \leqslant d$ for all $z \in \mathcal{A}_d(z_1, z_2)$. Such anticodes cannot have an arbitrary diameter: if $d(z_1, z_2) = \Delta$, then $\mathcal{A}_d(z_1, z_2) = \varnothing$ unless $d \geqslant \Delta$. For $d = \Delta$, it turns out that the optimal anticode $\mathcal{A}_\Delta(z_1, z_2)$ is the bounding rectangle of $z_1,\ z_2$.

Definition 3. Let $z_1 = (x_1, y_1), z_2 = (x_2, y_2), \ldots, z_n = (x_n, y_n)$ be n distinct points in $\mathbb{Z}^2$. The *bounding rectangle* of $z_1, z_2, \ldots, z_n$ is the smallest rectangle $\mathcal{R}(z_1, z_2, \ldots, z_n)$ with edges parallel to the axes that contains all the n points. Explicitly, let $x_{\max} = \max\{x_1, \ldots, x_n\}$, $x_{\min} = \min\{x_1, \ldots, x_n\}$, $y_{\max} = \max\{y_1, \ldots, y_n\}$, and $y_{\min} = \min\{y_1, \ldots, y_n\}$. Then

$$\mathcal{R}(z_1, z_2, \ldots, z_n) \stackrel{\text{def}}{=} \left\{(x, y) \in \mathbb{Z}^2 \ :\ x_{\min} \leqslant x \leqslant x_{\max} \ \text{ and } \ y_{\min} \leqslant y \leqslant y_{\max}\right\}. \tag{6}$$

By Theorem 1, the tristance of *any* three points in $\mathcal{R}(z_1, z_2)$ is at most $d(z_1, z_2)$. Moreover, if $z \notin \mathcal{R}(z_1, z_2)$, then $d_3(z_1, z_2, z) > d(z_1, z_2)$. This implies that if $d = d(z_1, z_2)$, then $\mathcal{R}(z_1, z_2)$ is the optimal anticode $\mathcal{A}_d(z_1, z_2)$. For general d, we have the following theorem:

Theorem 8. *Let $z_1 = (x_1, y_1), z_2 = (x_2, y_2)$ be distinct points in $\mathbb{Z}^2$ and assume, w.l.o.g., that $x_2 \geqslant x_1$ and $y_2 \geqslant y_1$ so that $d(z_1, z_2) = (x_2 - x_1) + (y_2 - y_1)$. Let $\mathcal{A}_d(z_1, z_2)$ be the optimal tristance anticode in $\mathcal{G}_2^{\boxplus}$ of diameter $d \geqslant d(z_1, z_2)$ centered about z_1 and z_2. Write $c = d - d(z_1, z_2)$. Then $\mathcal{A}_d(z_1, z_2)$ consists of all $z = (x, y)$ in $\mathbb{Z}^2$ such that*

$$x_1 - c \leqslant x \leqslant x_2 + c\,, \quad x_1 + y_1 - c \leqslant x + y \leqslant x_2 + y_2 + c, \tag{7}$$

$$y_1 - c \leqslant y \leqslant y_2 + c\,, \quad x_1 - y_2 - c \leqslant x - y \leqslant x_2 - y_1 + c. \tag{8}$$

Proof. The given points z_1, z_2 completely determine all the other points in $\mathcal{A}_d(z_1, z_2)$, as follows: $\mathcal{A}_d(z_1, z_2) = \{z \in \mathbb{Z}^2 : d_3(z_1, z_2, z) \leqslant d\}$. It is now easy to see that $z \in \mathcal{A}_d(z_1, z_2)$ if and only if the L_1-distance from z to (the closest point of) the bounding rectangle $\mathcal{R}(z_1, z_2)$ is at most $c = d - d(z_1, z_2)$. This is precisely the property expressed by (7) and (8). □

2.3. General tristance anticodes in the grid graph

We will now use the results of §2.2, especially Theorem 8, to classify unrestricted (non-centered) optimal tristance anticodes in $\mathcal{G}_2^{\boxplus}$. The subset of $\mathbb{Z}^2$ defined by Eqs. (7), (8) in Theorem 8 is an example of a set we call an *octagon*. We formalize this as follows.

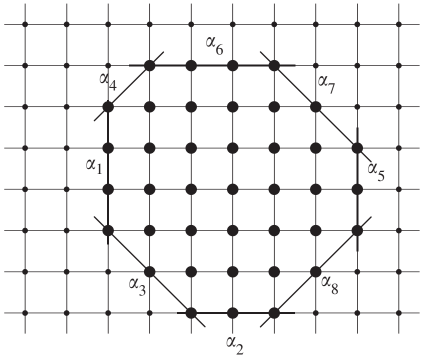

Fig. 2. A generic octagon $\mathcal{O}(\alpha_1, \alpha_2, \ldots, \alpha_8)$ in the grid graph $\mathcal{G}_2^{\boxplus}$.

Definition 4. Let $\alpha_1, \alpha_2, \ldots, \alpha_8$ be arbitrary real constants. An *octagon* $\mathcal{O}(\alpha_1, \alpha_2, \ldots, \alpha_8)$ is a subset of $\mathbb{Z}^2$ defined by the inequalities

$$\alpha_1 \leqslant x \leqslant \alpha_5, \qquad \alpha_3 \leqslant x + y \leqslant \alpha_7, \tag{9}$$

$$\alpha_2 \leqslant y \leqslant \alpha_6, \qquad \alpha_4 \leqslant x - y \leqslant \alpha_8. \tag{10}$$

A generic octagon $\mathcal{O}(\alpha_1, \alpha_2, \ldots, \alpha_8)$ is illustrated in Fig. 2. Note that $\mathcal{O}(\alpha_1, \alpha_2, \ldots, \alpha_8)$ may have fewer than eight sides (say, if $\alpha_3 \leqslant \alpha_1 + \alpha_2$), or may be empty altogether. Octagons will play an important role in this paper. Note that the L_1-spheres $\mathcal{S}_{d/2}(z_0)$ and $\mathcal{S}_{d/2}(z_0 + \xi)$ in Theorem 7 are octagons. By Theorem 8, the optimal anticode $\mathcal{A}_d(z_1, z_2)$ is also an octagon. The following lemma establishes another useful property of octagons.

Lemma 9. *The intersection of any two octagons is an octagon.*

Proof. It is clear that $\mathcal{O}(\alpha_1, \alpha_2, \ldots, \alpha_8) \cap \mathcal{O}(\beta_1, \beta_2, \ldots, \beta_8) = \mathcal{O}(\gamma_1, \gamma_2, \ldots, \gamma_8)$, where $\gamma_i = \max\{\alpha_i, \beta_i\}$ for $i = 1, 2, 3, 4$ and $\gamma_i = \min\{\alpha_i, \beta_i\}$ for $i = 5, 6, 7, 8$. $\square$

We are now in a position to begin the classification of unrestricted optimal tristance anticodes in $\mathcal{G}_2^{\boxplus}$. The next lemma establishes a certain closure property of such anticodes.

Lemma 10. *Let $\mathcal{A}_d$ be an optimal tristance anticode of diameter d in $\mathcal{G}_2^{\boxplus}$. Then $\mathcal{A}_d$ is closed under intersection with anticodes centered about pairs of its own points, namely*

$$\mathcal{A}_d = \bigcap_{z_1, z_2 \in \mathcal{A}_d} \mathcal{A}_d(z_1, z_2). \tag{11}$$

Proof. If $z \in \mathcal{A}_d$ then, by the definition of an anticode, we have $d_3(z_1, z_2, z) \leqslant d$ for all $z_1, z_2 \in \mathcal{A}_d$. Thus $z \in \mathcal{A}_d(z_1, z_2)$ for all $z_1, z_2 \in \mathcal{A}_d$, and hence $z \in \bigcap_{z_1, z_2 \in \mathcal{A}_d} \mathcal{A}_d(z_1, z_2)$.

Thus for *any* tristance anticode $\mathcal{A}$ of diameter d, we have

$$\mathcal{A} \subseteq \bigcap_{z_1, z_2 \in \mathcal{A}} \mathcal{A}_d(z_1, z_2).$$

Now, if $\mathcal{A}_d$ is optimal and $z \notin \mathcal{A}_d$, then there exist $z_1, z_2 \in \mathcal{A}_d$ such that $d_3(z_1, z_2, z) > d$; otherwise, we could adjoin z to $\mathcal{A}_d$ to obtain a larger anticode. For these $z_1, z_2 \in \mathcal{A}_d$, we have $z \notin \mathcal{A}_d(z_1, z_2)$. Hence $z \notin \bigcap_{z_1, z_2 \in \mathcal{A}_d} \mathcal{A}_d(z_1, z_2)$, and the lemma follows. □

Combining Theorem 8, Lemma 9 and Lemma 10 makes it possible to determine the *shape* of optimal tristance anticodes in the grid graph.

Lemma 11. *Let $\mathcal{A}_d$ be an optimal tristance anticode of diameter d in $\mathcal{G}_2^{\boxplus}$. Then $\mathcal{A}_d$ is an octagon $\mathcal{O}(\alpha_1, \alpha_2, \ldots, \alpha_8)$ for some $\alpha_1, \alpha_2, \ldots, \alpha_8 \in \mathbb{Z}$.*

Proof. By Lemma 10, we have $\mathcal{A}_d = \bigcap_{z_1, z_2 \in \mathcal{A}_d} \mathcal{A}_d(z_1, z_2)$. By Theorem 8, each of the sets $\mathcal{A}_d(z_1, z_2)$ is an octagon. By Lemma 9, an intersection of octagons is also an octagon. □

Using translations in $\mathbb{Z}^2$, we may always assume w.l.o.g. that $\alpha_1 = \alpha_2 = 0$ in (9) and (10). Thus, in view of Lemma 11, we have $\mathcal{A}_d = \mathcal{O}(0, 0, \alpha_3, \alpha_4, \ldots, \alpha_8)$, and it remains to determine the six integer parameters $\alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8$ as a function of the diameter d.

To this end, we first rewrite the definition of an octagon $\mathcal{O}(0, 0, \alpha_3, \alpha_4, \ldots, \alpha_8)$ in a different form. This octagon can be defined as the set of all $(x, y) \in \mathbb{Z}^2$ such that

$$0 \leqslant x \leqslant a, \quad c_0 \leqslant x + y \leqslant a + b - c_2, \tag{12}$$

$$0 \leqslant y \leqslant b, \quad c_3 - b \leqslant x - y \leqslant a - c_1, \tag{13}$$

where $a = \alpha_5$, $b = \alpha_6$, $c_0 = \alpha_3$, $c_1 = \alpha_5 - \alpha_8$, $c_2 = \alpha_5 + \alpha_6 - \alpha_7$, and $c_3 = \alpha_4 + \alpha_6$. We omit the tedious, but easy, proof of the transformation from (9)–(10) to (12)–(13).

We will use $\mathcal{O}(a, b, c_0, c_1, c_2, c_3)$ to denote an octagon $\mathcal{O}(0, 0, \alpha_3, \alpha_4, \ldots, \alpha_8)$ specified in terms of the alternative parameters a, b, c_1, c_2, c_3, c_4 of (12) and (13). It is easy to see from Fig. 3 that the size of $\mathcal{O}(a, b, c_0, c_1, c_2, c_3)$ is given by

$$|\mathcal{O}(a, b, c_0, c_1, c_2, c_3)| = (a+1)(b+1) - \sum_{i=0}^{3} \frac{c_i(c_i+1)}{2}. \tag{14}$$

The next step is to determine the maximum tristance d of an optimal tristance anticode $\mathcal{A}_d = \mathcal{O}(a, b, c_0, c_1, c_2, c_3)$ as a function of its parameters. We say that points z_1, z_2, z_3 in $\mathcal{A}_d$ are *diametric* if they attain the maximum tristance in $\mathcal{A}_d$, that is, if $d_3(z_1, z_2, z_3) = d$.

Lemma 12. *An optimal tristance anticode $\mathcal{A}_d = \mathcal{O}(a, b, c_0, c_1, c_2, c_3)$ always contains diametric points z_1^*, z_2^*, z_3^* such that $z_1^* = (x_1^*, 0)$ and $z_3^* = (x_3^*, b)$ for some $x_1^*, x_3^* \in \mathbb{Z}$.*

Proof. Let $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2)$, and $z_3 = (x_3, y_3)$ be three diametric points in $\mathcal{A}_d$. W.l.o.g. assume that $\min\{y_1, y_2, y_3\} = y_1$ and $\max\{y_1, y_2, y_3\} = y_3$. As $d_3(z_1, z_2, z_3) = d$,

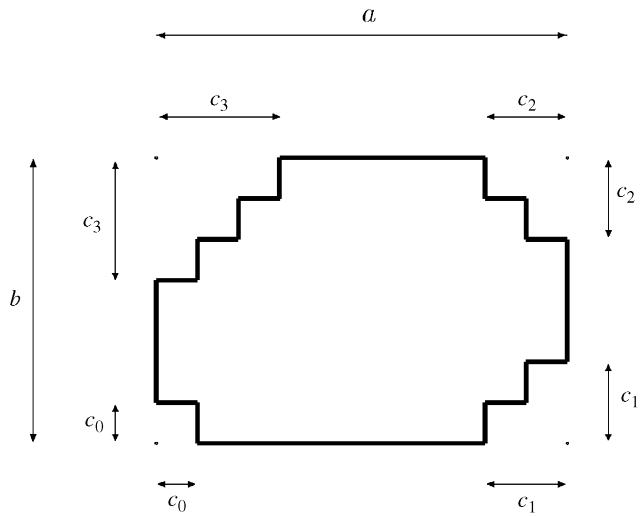

Fig. 3. An octagon $\mathcal{O}(0, 0, \alpha_3, \alpha_4, \ldots, \alpha_8)$ defined in terms of a, b, c_0, c_1, c_2, and c_3.

Theorem 1 implies that the point $z_1 - (0, 1) = (x_1, y_1 - 1)$ is not in $\mathcal{A}_d$. Referring to (9)–(13) along with Figs. 2 and 3, this in turn implies that either $y_1 = 0$ or $x_1 + y_1 = c_0$ or $x_1 - y_1 = a - c_1$. If $y_1 = 0$, we take $z_1^* = z_1$. Otherwise, if $x_1 + y_1 = c_0$, let

$$z_1' \stackrel{\text{def}}{=} z_1 - (x_1 - c_0, y_1) = (c_0, 0)$$

(the point z_1' is obtained from z_1 by moving down along the South–West edge of the octagon until reaching the South edge). Clearly $z_1' \in \mathcal{A}_d$. Moreover $d_3(z_1', z_2, z_3) \geqslant d_3(z_1, z_2, z_3) = d$, since replacing $y_1 = \min\{y_1, y_2, y_3\}$ by 0 increases the tristance by y_1 (cf. Theorem 1) while replacing x_1 by $c_0 = x_1 + y_1$ decreases the tristance by at most y_1. Hence the points z_1', z_2, z_3 are diametric, and we take $z_1^* = z_1'$. If $x_1 - y_1 = a - c_1$, we replace z_1 by $z_1' = (a - c_1, 0)$. Again, it is easy to see that z_1', z_2, z_3 are diametric, and we take $z_1^* = z_1'$. Now proceed with the diametric points z_1^*, z_2, z_3 in a similar fashion to obtain z_3^*. $\square$

Corollary 13. *The diameter of an optimal tristance anticode* $\mathcal{A}_d = \mathcal{O}(a, b, c_0, c_1, c_2, c_3)$ *is given by*

$$d = a + b - \min\{c_0, c_1, c_2, c_3\}. \tag{15}$$

Proof. W.l.o.g. assume that $\min\{c_0, c_1, c_2, c_3\} = c_0$. Let $z_1 = (x_1, 0)$, $z_2 = (x_2, y_2)$, and $z_3 = (x_3, b)$ be the diametric triple exhibited in Lemma 12. Then, in view of Theorem 1,

$$d = d_3(z_1, z_2, z_3) = b + \max\{x_1, x_2, x_3\} - \min\{x_1, x_2, x_3\}. \tag{16}$$

It is easy to see that $\max\{x_1, x_2, x_3\} - \min\{x_1, x_2, x_3\} \leqslant a - \min\{c_0, c_1, c_2, c_3\} = a - c_0$. Equality in this bound is achieved for $x_2 = a$ and $x_1 = c_0$, as illustrated in Fig. 4. $\square$

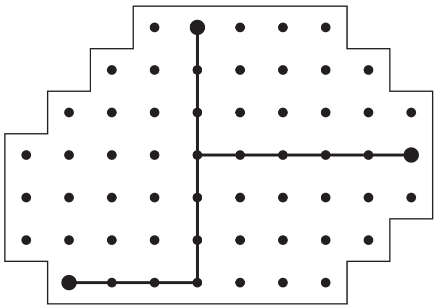

Fig. 4. A diametric configuration in $\mathcal{A}_d$.

Table 1
Parameters of optimal tristance anticodes in the grid graph $\mathcal{G}_2^{\boxplus}$

$d \pmod 7$	a	b	c	$\lvert\mathcal{A}_d\rvert$
0	$\frac{4d}{7}$	$\frac{4d}{7}$	$\frac{d}{7}$	$\frac{2d^2+6d+7}{7}$
1	$\frac{4d+3}{7}$	$\frac{4d-4}{7}$	$\frac{d-1}{7}$	$\frac{2d^2+6d+6}{7}$
2	$\frac{4d-1}{7}$	$\frac{4d-1}{7}$	$\frac{d-2}{7}$	$\frac{2d^2+6d+8}{7}$
3	$\frac{4d+2}{7}$	$\frac{4d-5}{7}$	$\frac{d-3}{7}$	$\frac{2d^2+6d+6}{7}$
4	$\frac{4d-2}{7}$	$\frac{4d-2}{7}$	$\frac{d-4}{7}$	$\frac{2d^2+6d+7}{7}$
5	$\frac{4d+1}{7}$	$\frac{4d+1}{7}$	$\frac{d+2}{7}$	$\frac{2d^2+6d+4}{7}$
	$\frac{4d+1}{7}$	$\frac{4d-6}{7}$	$\frac{d-5}{7}$	
6	$\frac{4d-3}{7}$	$\frac{4d-3}{7}$	$\frac{d-6}{7}$	$\frac{2d^2+6d+4}{7}$
	$\frac{4d+4}{7}$	$\frac{4d-3}{7}$	$\frac{d+1}{7}$	

Corollary 14. *If $\mathcal{A}_d = \mathcal{O}(a, b, c_0, c_1, c_2, c_3)$ is an optimal tristance anticode, then*

$$c_0 = c_1 = c_2 = c_3. \tag{17}$$

Proof. Obviously, (17) maximizes the size of $\mathcal{A}_d = \mathcal{O}(a, b, c_0, c_1, c_2, c_3)$ in (14) for the given diameter $d = a + b - \min\{c_0, c_1, c_2, c_3\}$. □

Theorem 15. *Let $\mathcal{A}_d$ be an optimal tristance anticode of diameter d in the grid graph $\mathcal{G}_2^{\boxplus}$. Then*

$$|\mathcal{A}_d| = \left\lceil \frac{2d^2 + 6d + 4}{7} \right\rceil = \left\lceil \frac{2(d+1)(d+2)}{7} \right\rceil.$$

Moreover, up to rotation by an angle of $\pi/2$ and translation, $\mathcal{A}_d = \mathcal{O}(a, b, c, c, c, c)$ where the parameters a, b, and c are given as a function of d in Table 1.

Proof. It follows from Lemma 11 in conjunction with Corollary 14 that $\mathcal{A}_d$ is an octagon of the form $\mathcal{O}(a, b, c, c, c, c)$, for some $a, b, c \in \mathbb{Z}$. The size of $\mathcal{A}_d$ is $(a+1)(b+1) - 2c(c+1)$

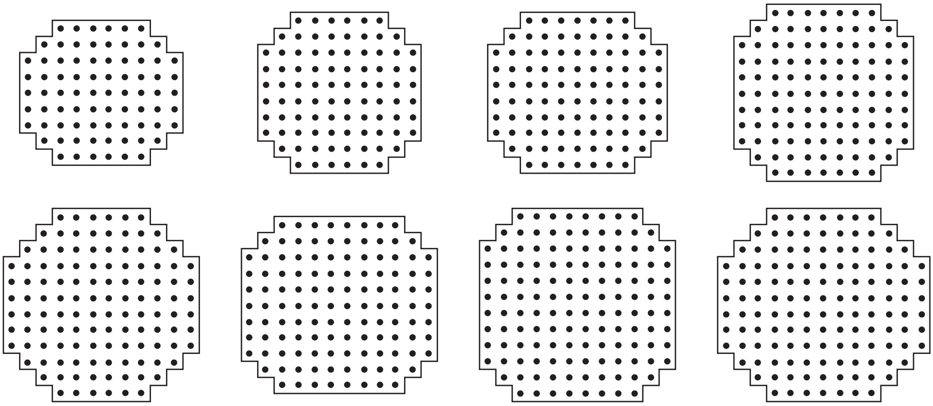

Fig. 5. Optimal tristance anticodes in $\mathcal{G}_2^{\boxplus}$ of diameter $d = 15, 16, \ldots, 20$.

by (14) and its diameter is $d = a + b - c$ by Corollary 13. To complete the proof, it remains to maximize $(a+1)(b+1) - 2c(c+1)$ subject to the constraint $a + b - c = d$. The solution to this simple optimization problem is given in Table 1. $\square$

Theorem 15 completely characterizes the optimal tristance anticodes of a given diameter in the grid graph $\mathcal{G}_2^{\boxplus}$. Some examples of such anticodes are illustrated in Fig. 5.

3. Optimal tristance anticodes in the infinity graph and the hexagonal graph models

We now classify the optimal tristance anticodes in two related graphical models: the infinity graph $\mathcal{G}_2^{\infty}$ and the hexagonal graph $\mathcal{G}_2^{\hexagon}$ (defined in §1). To obtain the classification for $\mathcal{G}_2^{\infty}$, we make use of a mapping $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ which takes $\mathcal{G}_2^{\infty}$ into a power graph of $\mathcal{G}_2^{\boxplus}$. For the hexagonal graph $\mathcal{G}_2^{\hexagon}$, we first derive an expression for the corresponding tristance $d_3^{\hexagon}(\cdot)$, and then follow the same line of argument as in the previous section.

3.1. Optimal tristance anticodes in the infinity graph

It is easy to see that the unique, up to translation, optimal anticode of diameter d in $\mathcal{G}_2^{\infty}$ is the square $\mathcal{S}_d = \{(x, y) \in \mathbb{Z}^2 : 0 \leqslant x \leqslant d \text{ and } 0 \leqslant y \leqslant d\}$. To deal with tristance anticodes, we first need an expression for tristance in $\mathcal{G}_2^{\infty}$. It turns out that tristance in $\mathcal{G}_2^{\infty}$ is related to tristance in $\mathcal{G}_2^{\boxplus}$ via the mappings $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ and $\varphi \in v : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$\varphi(x, y) = (x - y, x + y) \quad \text{and} \quad \varphi \in v(x, y) = \left(\frac{x + y}{2}, \frac{y - x}{2}\right). \tag{18}$$

Geometrically, the mapping φ is simply a rotation by an angle of $\pi/4$ followed by scaling by a factor of $\sqrt{2}$. Note that $\varphi(\mathbb{Z}^2) = D_2$, where $D_2 = \{(x, y) \in \mathbb{Z}^2 : x + y \equiv 0 \bmod 2\}$

is the two-dimensional checkerboard lattice. Also note that for all $z_1, z_2 \in \mathbb{Z}^2$, we have

$$d^\infty(z_1, z_2) = \frac{|(x_1 - y_1) - (x_2 - y_2)|}{2} + \frac{|(x_1 + y_1) - (x_2 + y_2)|}{2} = \frac{d\big(\varphi(z_1), \varphi(z_2)\big)}{2}. \tag{19}$$

Now define the graph $\varphi(\mathcal{G}_2^\infty) = (V, E)$ as follows: $V = \varphi(\mathbb{Z}^2) = D_2$ and $\{z_1, z_2\} \in E$ if and only if $\{\varphi \in v(z_1), \varphi \in v(z_2)\}$ is an edge in $\mathcal{G}_2^\infty$. Clearly, the graphs $\mathcal{G}_2^\infty$ and $\varphi(\mathcal{G}_2^\infty)$ are isomorphic. It follows from (19) that the edges of $\varphi(\mathcal{G}_2^\infty)$ are precisely the paths of length 2 in the grid graph $\mathcal{G}_2^\boxplus$. This fact was used in [9] to prove the following theorem:

Theorem 16. *Let $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2)$, $z_3 = (x_3, y_3)$ be three distinct points in $\mathbb{Z}^2$, and let $z_1', z_2', z_3' \in D_2$ denote their images under φ. Then*

$$d_3^\infty(z_1, z_2, z_3) = \big\lceil d_3(z_1', z_2', z_3')/2 \big\rceil.$$

Theorem 16 and the mappings in (18) make it possible to classify the optimal tristance anticodes in $\mathcal{G}_2^\infty$ using the classification of tristance anticodes in $\mathcal{G}_2^\boxplus$ carried out in §2.

Theorem 17. *Let $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2)$ be distinct points in $\mathbb{Z}^2$ and assume, w.l.o.g., that $x_2 - x_1 \geqslant |y_2 - y_1|$ so that $d^\infty(z_1, z_2) = x_2 - x_1$. Let $\mathcal{A}_d^\infty(z_1, z_2)$ be the optimal tristance anticode in $\mathcal{G}_2^\infty$ of diameter $d \geqslant d^\infty(z_1, z_2)$ centered about z_1 and z_2. Let $c = d - d^\infty(z_1, z_2)$. Then $\mathcal{A}_d^\infty(z_1, z_2)$ consists of all $z = (x, y)$ in $\mathbb{Z}^2$ such that*

$$x_1 - c \leqslant x \leqslant x_2 + c, \tag{20}$$

$$x_1 + y_1 - 2c \leqslant x + y \leqslant x_2 + y_2 + 2c, \tag{21}$$

$$x_1 - y_1 - 2c \leqslant x - y \leqslant x_2 - y_2 + 2c, \tag{22}$$

$$(x_1 + y_1) - (x_2 - y_2) - 2c \leqslant 2y \leqslant (x_2 + y_2) - (x_1 - y_1) + 2c. \tag{23}$$

Proof. Let $z_1', z_2' \in D_2$ be the images of z_1 and z_2 under φ. Let $z \in \mathcal{A}_d^\infty(z_1, z_2)$. Then $d_3^\infty(z_1, z_2, z) \leqslant d$ and $d_3(z_1', z_2', \varphi(z)) \leqslant 2d$ by Theorem 16, so that $\varphi(z) \in \mathcal{A}_{2d}(z_1', z_2')$. Since $\varphi(z) \in D_2$ for all $z \in \mathbb{Z}^2$, it follows that

$$\varphi\big(\mathcal{A}_d^\infty(z_1, z_2)\big) \subseteq \mathcal{A}_{2d}(z_1', z_2') \cap D_2.$$

Conversely, let $z' \in \mathcal{A}_{2d}(z_1', z_2') \cap D_2$. Then $d_3(z_1', z_2', z') \leqslant 2d$ and $d_3^\infty(z_1, z_2, \varphi \in v(z')) \leqslant d$ by Theorem 16. Hence $\varphi \in v\big(\mathcal{A}_{2d}(z_1', z_2') \cap D_2\big) \subseteq \mathcal{A}_d^\infty(z_1, z_2)$ and therefore

$$\varphi\big(\mathcal{A}_d^\infty(z_1, z_2)\big) = \mathcal{A}_{2d}(z_1', z_2') \cap D_2. \tag{24}$$

In view of (24) and Theorem 8, a point $z \in \mathbb{Z}^2$ belongs to $\mathcal{A}_d^\infty(z_1, z_2)$ if and only if $\varphi(z)$ satisfies conditions (7)–(8) for $z_1' = \varphi(z_1)$ and $z_2' = \varphi(z_2)$, with c replaced by $2d - 2d_3^\infty(z_1, z_2)$ in view of (19). This is precisely the property expressed by (20)–(23). $\square$

Corollary 18. *Let $\mathcal{A}_d^\infty$ be an optimal tristance anticode of diameter d in $\mathcal{G}_2^\infty$. Then $\mathcal{A}_d^\infty$ is an octagon $\mathcal{O}(\alpha_1, \alpha_2, \ldots, \alpha_8)$ for some $\alpha_1, \alpha_2, \ldots, \alpha_8 \in \mathbb{Z}$.*

Proof. It is obvious from (20)–(23) that the set $\mathcal{A}_d^\infty(z_1, z_2)$ is an octagon. Since the closure property of Lemma 10 holds regardless of a particular distance model, the corollary now follows in exactly the same way as Lemma 11. □

As before, we can use translations in $\mathbb{Z}^2$ to write the octagon $\mathcal{A}_d^\infty$ as $\mathcal{O}(a, b, c_0, c_1, c_2, c_3)$ for some $a, b, c_i \in \mathbb{Z}$. Then the cardinality of $\mathcal{A}_d^\infty$ is given by (14), and the next step is to determine its diameter d as a function of the parameters a, b, c_0, c_1, c_2, c_3.

Lemma 19. *The diameter of an optimal tristance anticode $\mathcal{A}_d^\infty = \mathcal{O}(a, b, c_0, c_1, c_2, c_3)$ in the infinity graph $\mathcal{G}_2^\infty$ is given by*

$$d = a + b - \left\lfloor \frac{\min\{a + c_0 + c_1, a + c_2 + c_3, b + c_0 + c_3, b + c_1 + c_2\}}{2} \right\rfloor. \tag{25}$$

Proof. We again make use of the mapping in (18), in conjunction with Corollary 13. First consider the set $\varphi(\mathcal{A}_d^\infty)$. Even though $\mathcal{A}_d^\infty = \mathcal{O}(a, b, c_0, c_1, c_2, c_3)$, the set $\varphi(\mathcal{A}_d^\infty)$ is, in general, not an octagon, since $\varphi(\mathcal{A}_d^\infty) \subset D_2$. However, we can convert this into an octagon by adjoining the "missing" points as follows:

$$\mathcal{A}' \stackrel{\text{def}}{=} \varphi(\mathcal{A}_d^\infty) \cup \left\{ z \in \mathbb{Z}^2 : \text{at least 3 of the 4 neighbors of } z \text{ in } \mathcal{G}_2^\boxplus \text{ are in } \varphi(\mathcal{A}_d^\infty) \right\}. \tag{26}$$

A straightforward analysis of the effect of the mapping φ on (12)–(13) now shows that $\mathcal{A}'$ is an octagon $(c_3 - b, c_0) + \mathcal{O}(a', b', c_0', c_1', c_2', c_3')$, where

$$a' = a + b - (c_1 + c_3), \quad c_0' = b - (c_0 + c_3), \quad c_1' = a - (c_0 + c_1), \tag{27}$$

$$b' = a + b - (c_0 + c_2), \quad c_2' = b - (c_1 + c_2), \quad c_3' = a - (c_2 + c_3). \tag{28}$$

Let d' denote the diameter of $\mathcal{A}'$, and let z_1, z_2, z_3 be diametric points in $\mathcal{A}_d^\infty$. Then $\varphi(z_1)$, $\varphi(z_2)$, $\varphi(z_3)$ are in $\mathcal{A}'$ and their tristance in $\mathcal{G}_2^\boxplus$ is at least $2d - 1$ by Theorem 16. Hence $d' \geqslant 2d - 1$. Now let z_1', z_2', z_3' be diametric points in $\mathcal{A}'$. If $z_1' \notin \varphi(\mathcal{A}_d^\infty)$, then it has at least three neighbors in $\varphi(\mathcal{A}_d^\infty)$ by (26). By Theorem 1, this means that we can replace z_1', z_2', z_3' by another diametric configuration $z_1'', z_2', z_3' \in \mathcal{A}'$, where z_1'' is a neighbor of z_1' such that $z_1'' \in \varphi(\mathcal{A}_d^\infty)$. Repeating the argument for z_2' and z_3', we can find points $z_1', z_2', z_3' \in \varphi(\mathcal{A}_d^\infty)$ such that $d_3(z_1', z_2', z_3') = d'$. We now have $d_3^\infty(\varphi \in v(z_1'), \varphi \in v(z_2'), \varphi \in v(z_2')) = \lceil d'/2 \rceil \leqslant d$, in view of Theorem 16. Hence $d = \lceil d'/2 \rceil$. But $d' = a' + b' - \min\{c_0', c_1', c_2', c_3'\}$ by Corollary 13. The lemma now follows straightforwardly from (27)–(28). □

Theorem 20. *Let $\mathcal{A}_d^\infty$ be an optimal tristance anticode of diameter d in the infinity graph. Then*

$$|\mathcal{A}_d^\infty| = \left\lceil \frac{4d^2 + 8d + 2}{7} \right\rceil.$$

Moreover, up to rotation by an angle of $\pi/2$ and translation, $\mathcal{A}_d^\infty = \mathcal{O}(a, b, c_0, c_1, c_2, c_3)$ where the parameters a, b, c_0, c_1, c_2, c_3 are given as a function of d in Table 2.

Table 2
Parameters of optimal tristance anticodes in the infinity graph $\mathcal{G}_2^\infty$

d (mod 7)	a, b	c_0, c_2	c_1, c_3	$\lvert\mathcal{A}_d^\infty\rvert$
0	$\frac{6d}{7}$	$\frac{2d}{7}$	$\frac{2d}{7}$	$\frac{4d^2+8d+7}{7}$
1	$\frac{6d+1}{7}$	$\frac{2d-2}{7}$	$\frac{2d+5}{7}$	$\frac{4d^2+8d+2}{7}$
2	$\frac{6d+2}{7}$	$\frac{2d+3}{7}$	$\frac{2d+3}{7}$	$\frac{4d^2+8d+3}{7}$
3	$\frac{6d-4}{7}$	$\frac{2d-6}{7}$	$\frac{2d-6}{7}$	$\frac{4d^2+8d+3}{7}$
4	$\frac{6d-3}{7}$	$\frac{2d-8}{7}$	$\frac{2d-1}{7}$	$\frac{4d^2+8d+2}{7}$
5	$\frac{6d-2}{7}$	$\frac{2d-3}{7}$	$\frac{2d-3}{7}$	$\frac{4d^2+8d+7}{7}$
6	$\frac{6d-1}{7}$	$\frac{2d-5}{7}$	$\frac{2d+2}{7}$	$\frac{4d^2+8d+4}{7}$

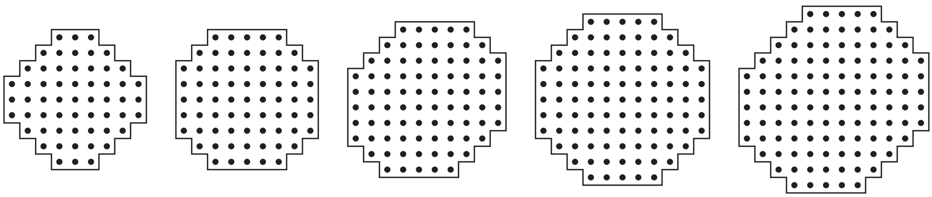

Fig. 6. Optimal tristance anticodes in $\mathcal{G}_2^\infty$ of diameter $d = 9, 10, \ldots, 13$.

Proof. In view of Corollary 18 and Lemma 19, we need to maximize the cardinality of $\mathcal{A}_d^\infty$ given by (14) subject to constraint (25). Let $t = a + b - 2d + \max\{a, b\}$. If t is even, then choosing $c_0 = c_1 = c_2 = c_3 = t/2$ satisfies (25) and maximizes (14). If t is odd, then the corresponding extremal values are $c_0 = c_2 = (t-1)/2$ and $c_1 = c_3 = (t+1)/2$ (or vice versa). If we now assume w.l.o.g. that $a \geqslant b$, then the cardinality of $\mathcal{A}_d^\infty$ is given by $(a+1)(b+1) - \lfloor (2a + b - 2d + 1)^2/2 \rfloor$. It remains to maximize this expression, subject to $a \geqslant b$. The solution to this optimization problem is given in Table 2. $\square$

Theorem 20 completes our classification of optimal tristance anticodes in the infinity graph $\mathcal{G}_2^\infty$. Some examples of such anticodes are illustrated in Fig. 6.

3.2. *Optimal tristance anticodes in the hexagonal graph*

Many different coordinate systems for the hexagonal lattice A_2 are known [7]. For our purposes, it would be most convenient to identify A_2 with the Eisenstein integers. That is, we write $A_2 = \{x + \omega y \;:\; x, y \in \mathbb{Z}\}$, where $\omega = -1/2 + \sqrt{3}/2\, i$ is a complex cube root of unity.[2] Thus a generic vertex v of $\mathcal{G}_2^{\hexagon}$ will be written as $v = (x, y)$, with the understanding that $v = x + \omega y$. The resulting labeling of the hexagonal graph is shown in Fig. 7.

[2] Note that the numbers 1, ω, and ω^2 represent the three different edge orientations in the hexagonal graph (cf. Fig. 7). However, two rather than three coordinates suffice to describe $\mathcal{G}_2^{\hexagon}$, since $\omega^2 = \bar{\omega} = -1 - \omega$.

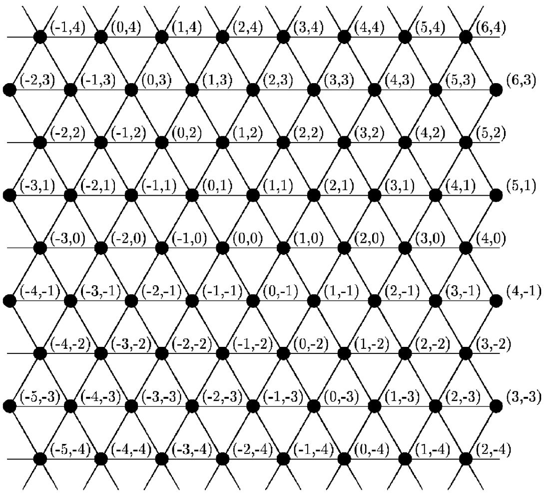

Fig. 7. The graph $\mathcal{G}_2^{\hexagon}$ with vertices labeled by the Eisenstein integers.

Our first task is to find expressions for distance and tristance in $\mathcal{G}_2^{\hexagon}$. To this end, let us introduce the following notation: given $a, b \in \mathbb{Z}$, we shall write $\overline{\max}\{a, b\} = \max\{a, b, 0\}$ and $\underline{\min}\{a, b\} = \min\{a, b, 0\}$. Now let $v_1 = (x_1, y_1)$ and $v_2 = (x_2, y_2)$ be two arbitrary points of A_2. It is easy to see that the distance between v_1 and v_2 in $\mathcal{G}_2^{\hexagon}$ is given by

$$d^{\hexagon}(v_1, v_2) \;=\; \overline{\max}\,\{x_1 - x_2, y_1 - y_2\} \;-\; \underline{\min}\,\{x_1 - x_2, y_1 - y_2\}. \tag{29}$$

Note that if $(x_1-x_2)(y_1-y_2)\leqslant 0$, then (29) reduces to $d^{\hexagon}(v_1, v_2) = |x_1-x_2| + |y_1-y_2|$ while if $(x_1-x_2)(y_1-y_2)\geqslant 0$ then $d^{\hexagon}(v_1, v_2) = \max\{|x_1-x_2|, |y_1-y_2|\}$. Thus distance in $\mathcal{G}_2^{\hexagon}$ is, in a sense, "half-way" between the L_1-distance of $\mathcal{G}_2^{\boxplus}$ and the L_∞-distance of $\mathcal{G}_2^{\infty}$. Deriving an expression for tristance in $\mathcal{G}_2^{\hexagon}$ is a bit more involved. First, we need a lemma.

Lemma 21. *Let v_1, v_2, and v_3 be distinct points in A_2. Then there exists a point $v \in A_2$, such that $d_3^{\hexagon}(v_1, v_2, v_3) = d^{\hexagon}(v_1, v) + d^{\hexagon}(v_2, v) + d^{\hexagon}(v_3, v)$.*

Proof. By definition, $d_3^{\hexagon}(v_1, v_2, v_3)$ is the number of edges in a minimal spanning tree for v_1, v_2, v_3 in the hexagonal graph $\mathcal{G}_2^{\hexagon}$. Let T be such a tree. Further, for $i = 1, 2, \ldots, 6$, let η_i denote the number of vertices of degree i in T. First observe that $\eta_1 \leqslant 3$. Indeed, if there is a leaf in T that is not one of v_1, v_2, v_3, then we could remove this leaf along with the single edge incident upon it, to obtain a smaller spanning tree for v_1, v_2, v_3 in $\mathcal{G}_2^{\hexagon}$. Now, the order of T is $|V| = \eta_1 + \eta_2 + \eta_3 + \eta_4 + \eta_5 + \eta_6$, while its size is $|E| = (\eta_1 + 2\eta_2 + 3\eta_3 + 4\eta_4 + 5\eta_5 + 6\eta_6)/2$. Since T is a tree, we have $|V| - |E| = 1$. With $|V|, |E|$ expressed in terms of $\eta_1, \eta_2, \ldots, \eta_6$, this condition is equivalent to

$$\eta_3 + 2\eta_4 + 3\eta_5 + 4\eta_6 \;=\; \eta_1 - 2 \;\leqslant\; 1.$$

It follows that $\eta_4 = \eta_5 = \eta_6 = 0$ and $\eta_3 \leqslant 1$. In other words, there are only two possible configurations for T: either it is *star-like*, with a single vertex of degree 3 and v_1, v_2, v_3 as its three leaves, or it is *snake-like* with only some two of v_1, v_2, v_3 as leaves and all other vertices of degree 2. If T is star-like, we take v to be the unique vertex of degree 3 in T. If T is snake-like, we take v to be one of v_1, v_2, v_3, the one which is not a leaf in T. $\square$

Theorem 22. *Let $v_1 = (x_1, y_1)$, $v_2 = (x_2, y_2)$, $v_3 = (x_3, y_3)$ be distinct points in A_2. Let x_{mid} denote the middle value among x_1, x_2, x_3—that is, if x'_1, x'_2, x'_3 is a permutation of x_1, x_2, x_3 such that $x'_1 \leqslant x'_2 \leqslant x'_3$, then $x_{\mathrm{mid}} = x'_2$. Let y_{mid} be similarly defined. Then*

$$d_3^{\hexagon}(v_1, v_2, v_3) = \sum_{i=1}^{3} \Big(\overline{\max}\{x_i - x_{\mathrm{mid}}, y_i - y_{\mathrm{mid}}\} - \underline{\min}\{x_i - x_{\mathrm{mid}}, y_i - y_{\mathrm{mid}}\} \Big). \quad (30)$$

Proof. Let $v = (x, y)$ be a point with $d_3^{\hexagon}(v_1, v_2, v_3) = d^{\hexagon}(v_1, v) + d^{\hexagon}(v_2, v) + d^{\hexagon}(v_3, v)$. Such a point exists by Lemma 21. Then by (29) we have

$$d_3^{\hexagon}(v_1, v_2, v_3) = \sum_{i=1}^{3} \Big(\overline{\max}\{x_i - x, \ y_i - y\} - \underline{\min}\{x_i - x, \ y_i - y\} \Big). \quad (31)$$

Clearly, the expression in (31) is an upper bound on $d_3^{\hexagon}(v_1, v_2, v_3)$ for all $x, y \in \mathbb{Z}$. To establish the tristance, it remains to find $x, y \in \mathbb{Z}$ that minimize this expression. This is a tedious, but simple, optimization problem (an optimal solution must satisfy $x \in \{x_1, x_2, x_3\}$ and $y \in \{y_1, y_2, y_3\}$, so there are nine cases to consider). The reader can easily verify that $x = x_{\mathrm{mid}}$ and $y = y_{\mathrm{mid}}$ is indeed an optimal solution. $\square$

Remark. The expressions $\max\{\cdot, \cdot, 0\}$ and $\min\{\cdot, \cdot, 0\}$ in (29), (30) arise from the asymmetry in our coordinate system for $\mathcal{G}_2^{\hexagon}$. If, instead, we represent a generic point of A_2 as $v = (x, y, z)$, with the understanding that $v = x + \omega y + \omega^2 z$, then (29) becomes

$$d^{\hexagon}(v_1, v_2) = \max\{x_1 - x_2, y_1 - y_2, z_1 - z_2\} - \min\{x_1 - x_2, y_1 - y_2, z_1 - z_2\}$$

and (30) should be modified accordingly. In some sense, these expressions are more natural, since they reflect the three edge directions in $\mathcal{G}_2^{\hexagon}$. On the other hand, this coordinate system is redundant: (x, y, z) and $(x - \delta, y - \delta, z - \delta)$ represent the same point of A_2 for all $\delta \in \mathbb{Z}$, since $1 + \omega + \omega^2 = 0$. One can use this property to zero-out any one of the three coordinates. Zeroing out the last coordinate by choosing $\delta = z$ (as we have done) is precisely the source for the remnant zeros in $\max\{\cdot, \cdot, 0\}$ and $\min\{\cdot, \cdot, 0\}$ in (29), (30).

From here, we proceed along the lines of §2.3. Let a *hexagon* $\mathscr{H}(\alpha_1, \alpha_2, \ldots, \alpha_6)$ be a subset of A_2 defined by the inequalities

$$\alpha_1 \leqslant x \leqslant \alpha_4, \qquad \alpha_2 \leqslant y \leqslant \alpha_5, \qquad \alpha_3 \leqslant x - y \leqslant \alpha_6. \quad (32)$$

The next lemma and theorem show that the optimal centered anticode $\mathcal{A}_d^{\hexagon}(v_1, v_2) \subset A_2$, centered about an arbitrary pair of points $v_1, v_2 \in A_2$, is a hexagon for all $d \geqslant d^{\hexagon}(v_1, v_2)$.

Lemma 23. *Let $v_1 = (x_1, y_1), v_2 = (x_2, y_2)$ be distinct points of A_2. Then $d_3^{\hexagon}(v_1, v_2, v_3) = d^{\hexagon}(v_1, v_2)$ if and only if v_3 belongs to the bounding parallelepiped of v_1, v_2, which is a subset of A_2 defined by the inequalities*

$$\min\{x_1, x_2\} \leqslant \quad x \quad \leqslant \max\{x_1, x_2\}, \tag{33}$$

$$\min\{y_1, y_2\} \leqslant \quad y \quad \leqslant \max\{y_1, y_2\}, \tag{34}$$

$$\min\{x_1 - y_1, x_2 - y_2\} \leqslant x - y \leqslant \max\{x_1 - y_1, x_2 - y_2\}. \tag{35}$$

Proof. Let $\mathcal{P}(v_1, v_2)$ denote the bounding parallelepiped of v_1, v_2 (note that it is, indeed, a parallelepiped since one of (33)–(35) is always redundant). Assume w.l.o.g. that $x_1 \leqslant x_2$. ($\Leftarrow$) Suppose $v_3 \in \mathcal{P}(v_1, v_2)$. Then (33), (34) imply that $x_{\mathrm{mid}} = x_3$ and $y_{\mathrm{mid}} = y_3$ in (30). Thus (30) reduces to

$$d_3^{\hexagon}(v_1, v_2, v_3) = \begin{cases} (y_1 - y_3) - (x_1 - x_3) + (x_2 - x_3) - (y_2 - y_3) & \text{if } y_1 \geqslant y_2, \\ \max\{x_2 - x_3, y_2 - y_3\} - \min\{x_1 - x_3, y_1 - y_3\} & \text{if } y_1 \leqslant y_2. \end{cases} \tag{36}$$

If $y_1 \geqslant y_2$, then $d_3^{\hexagon}(v_1, v_2, v_3) = d^{\hexagon}(v_1, v_2)$ directly by (36) and (29). If $y_1 \leqslant y_2$, then the inequality $d_3^{\hexagon}(v_1, v_2, v_3) \leqslant d^{\hexagon}(v_1, v_2)$ follows by straightforward manipulation from (36) and (35). ($\Rightarrow$) Now suppose that $d_3^{\hexagon}(v_1, v_2, v_3) = d^{\hexagon}(v_1, v_2)$. Then every minimal spanning tree for v_1, v_2, v_3 in $\mathcal{G}_2^{\hexagon}$ must be snake-like, with v_1, v_2 as its leaves. Indeed, by Lemma 21 we have

$$d_3^{\hexagon}(v_1, v_2, v_3) = d^{\hexagon}(v_1, v) + d^{\hexagon}(v_2, v) + d^{\hexagon}(v_3, v) = d^{\hexagon}(v_1, v_2)$$

for some $v \in A_2$. This is only possible if $d^{\hexagon}(v_3, v) = 0$, since $d^{\hexagon}(v_1, v) + d^{\hexagon}(v_2, v) \geqslant d^{\hexagon}(v_1, v_2)$ by the triangle inequality. The fact that $d^{\hexagon}(v_3, v) = 0$ implies that the third term in the summation of (30) is zero, which is only possible if $x_{\mathrm{mid}} = x_3$ and $y_{\mathrm{mid}} = y_3$. This establishes (33) and (34). Moreover, the expression for $d_3^{\hexagon}(v_1, v_2, v_3)$ in (30) once again reduces to (36). If $y_1 \leqslant y_2$, then $d_3^{\hexagon}(v_1, v_2, v_3) = d^{\hexagon}(v_1, v_2)$ further reduces to

$$\max\{x_2 - x_3, y_1 - y_3\} - \min\{x_1 - x_3, y_1 - y_3\} = \max\{x_2 - x_1, y_2 - y_1\}$$

by (36) and (29). It is straightforward to show that this condition is equivalent to (35). Otherwise, if $y_1 \geqslant y_2$, then (33) and (34) imply (35), and we are done. □

Theorem 24. *Let $v_1 = (x_1, y_1)$, $v_2 = (x_2, y_2)$ be distinct points of A_2, and let $\mathcal{A}_d^{\hexagon}(v_1, v_2)$ be the optimal tristance anticode in $\mathcal{G}_2^{\hexagon}$ of diameter $d \geqslant d^{\hexagon}(v_1, v_2)$ centered about v_1 and v_2. Write $c = d - d^{\hexagon}(v_1, v_2)$. Then $\mathcal{A}_d^{\hexagon}(v_1, v_2)$ consists of all $v = (x, y)$ in A_2 such that*

$$\min\{x_1, x_2\} - c \leqslant \quad x \quad \leqslant \max\{x_1, x_2\} + c, \tag{37}$$

$$\min\{y_1, y_2\} - c \leqslant \quad y \quad \leqslant \max\{y_1, y_2\} + c, \tag{38}$$

$$\min\{x_1 - y_1, x_2 - y_2\} - c \leqslant x - y \leqslant \max\{x_1 - y_1, x_2 - y_2\} + c. \tag{39}$$

Proof. When $c = 0$, the theorem follows immediately from Lemma 23. Otherwise, it is easy to see that $v \in \mathcal{A}_d^{\hexagon}(v_1, v_2)$ if and only if the distance in $\mathcal{G}_2^{\hexagon}$ from v to (the closest point of) the bounding parallelepiped $\mathcal{P}(v_1, v_2)$ is at most $c = d - d^{\hexagon}(v_1, v_2)$. This is precisely the property expressed by (37)–(39). $\square$

Corollary 25. *Let $\mathcal{A}_d^{\hexagon}$ be an optimal tristance anticode of diameter d in $\mathcal{G}_2^{\hexagon}$. Then $\mathcal{A}_d^{\hexagon}$ is a hexagon $\mathscr{H}(\alpha_1, \alpha_2, \ldots, \alpha_6)$ for some $\alpha_1, \alpha_2, \ldots, \alpha_6 \in \mathbb{Z}$.*

Proof. It is obvious from Theorem 24 and (32) that the set $\mathcal{A}_d^{\hexagon}(v_1, v_2)$ is a hexagon. The fact that the intersection of any two hexagons is a hexagon is also obvious (cf. Lemma 9). The corollary now follows in exactly the same way as Lemma 11 and Corollary 18. $\square$

Since A_2 is invariant under translation by a lattice point, we can again shift $\mathcal{A}_d^{\hexagon}$ to the origin, so that $\mathcal{A}_d^{\hexagon} = \mathscr{H}(0, 0, \alpha_3, \alpha_4, \alpha_5, \alpha_6)$, and then write it as $\mathscr{H}(a, b, c_1, c_3)$, where $a = \alpha_4, b = \alpha_5, c_1 = \alpha_4 - \alpha_6$, and $c_3 = \alpha_3 + \alpha_5$ (cf. Fig. 3). This is just a special case of (12)–(13).

Lemma 26. *The diameter of an optimal tristance anticode $\mathcal{A}_d^{\hexagon} = \mathscr{H}(a, b, c_1, c_3)$ in the hexagonal graph $\mathcal{G}_2^{\hexagon}$ is given by $d = a + b - \min\{c_1, c_3\}$.*

Proof. Assume w.l.o.g. that $c_1 \leqslant c_3$. Observe that we can further assume w.l.o.g. that $0 \leqslant c_1, c_3 \leqslant \min\{a, b\}$; otherwise at least one of the inequalities in

$$0 \leqslant x \leqslant a\,, \qquad 0 \leqslant y \leqslant b\,, \qquad c_3 - b \leqslant x - y \leqslant a - c_1 \tag{40}$$

is redundant, and the hexagon $\mathscr{H}(a, b, c_1, c_3)$ can be translated and/or re-parametrized so that $0 \leqslant c_1, c_3 \leqslant \min\{a, b\}$ holds. Thus the points $v_1 = (0, 0)$, $v_2 = (a, c_1)$, and $v_3 = (a, b)$ belong to $\mathscr{H}(a, b, c_1, c_3)$ by (40), and $d_3^{\hexagon}(v_1, v_2, v_3) = a + b - c_1$ by (30). Now let v_1, v_2, v_3 be arbitrary points in $\mathscr{H}(a, b, c_1, c_3)$. We assume w.l.o.g. that $x_1 \leqslant x_2 \leqslant x_3$, and distinguish between six cases. In each case, we compute $d_3^{\hexagon}(v_1, v_2, v_3)$ using (30).

Case 1: $y_3 \leqslant y_2 \leqslant y_1$. Then $d_3^{\hexagon}(v_1, v_2, v_3) = (x_3 - x_1) + (y_1 - y_3)$.

Case 2: $y_3 \leqslant y_1 \leqslant y_2$. Then $d_3^{\hexagon}(v_1, v_2, v_3) = (x_3 - x_1) + (y_2 - y_3)$.

Case 3: $y_2 \leqslant y_3 \leqslant y_1$. Then $d_3^{\hexagon}(v_1, v_2, v_3) = (x_3 - x_1) + (y_1 - y_2)$.

Case 4: $y_2 \leqslant y_1 \leqslant y_3$. Then $d_3^{\hexagon}(v_1, v_2, v_3) = (x_2 - x_1) + (y_1 - y_2) + \max\{x_3 - x_2, y_3 - y_1\}$.

Case 5: $y_1 \leqslant y_3 \leqslant y_2$. Then $d_3^{\hexagon}(v_1, v_2, v_3) = (x_3 - x_2) + (y_2 - y_3) + \max\{x_2 - x_1, y_3 - y_1\}$.

Case 6: $y_1 \leqslant y_2 \leqslant y_3$. Then $d_3^{\hexagon}(v_1, v_2, v_3) =$
$\max\{x_3 - x_2, y_3 - y_2\} + \max\{x_2 - x_1, y_2 - y_1\}$.

In each of these cases, it is straightforward to show that $d_3^{\hexagon}(v_1, v_2, v_3) \leqslant a + b - c_3$ or $d_3^{\hexagon}(v_1, v_2, v_3) \leqslant a + b - c_1$ by (40), and the lemma follows. $\square$

Theorem 27. *Let $\mathcal{A}_d^{\hexagon}$ be an optimal tristance anticode of diameter d in the hexagonal graph. Then*

$$|\mathcal{A}_d^{\hexagon}| = \left\lceil \frac{d^2 + 3d + 2}{3} \right\rceil = \left\lceil \frac{(d+1)(d+2)}{3} \right\rceil.$$

Table 3
Parameters of optimal tristance anticodes in the hexagonal graph $\mathcal{G}_2^{\hexagon}$

$d \pmod 3$	a	b	c_1, c_3	$\lvert\mathcal{A}_d^{\hexagon}\rvert$
0	$\frac{2d}{3}$	$\frac{2d}{3}$	$\frac{d}{3}$	$\frac{d^2+3d+3}{3}$
1	$\frac{2d+1}{3}$	$\frac{2d+1}{3}$	$\frac{d+2}{3}$	$\frac{d^2+3d+2}{3}$
	$\frac{2d+1}{3}$	$\frac{2d-2}{3}$	$\frac{d-1}{3}$	
2	$\frac{2d-1}{3}$	$\frac{2d-1}{3}$	$\frac{d-2}{3}$	$\frac{d^2+3d+2}{3}$
	$\frac{2d+2}{3}$	$\frac{2d-1}{3}$	$\frac{d+1}{3}$	

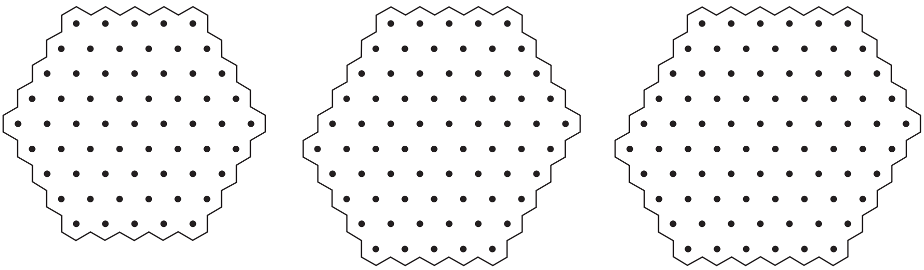

Fig. 8. Optimal tristance anticodes in $\mathcal{G}_2^{\hexagon}$ of diameter $d = 12, 13, 14$.

Moreover, up to rotation by an angle of $\pi/3$ *and translation,* $\mathcal{A}_d^{\hexagon} = \mathscr{H}(a, b, c_1, c_3)$ *where the parameters* a, b, c_1, c_3 *are given as a function of* d *in Table 3.*

Proof. The optimal tristance anticode is a hexagon $\mathscr{H}(a, b, c_1, c_3)$ by Corollary 25. Its cardinality is $|\mathcal{A}_d^{\hexagon}| = (a+1)(b+1) - \frac{1}{2}c_1(c_1+1) - \frac{1}{2}c_3(c_3+1)$ as in (14), and its diameter is $a + b - \min\{c_1, c_3\}$ by Lemma 26. Clearly, the choice $c_1 = c_3 = c$ maximizes $|\mathcal{A}_d^{\hexagon}|$ for a given diameter. It remains to maximize $(a+1)(b+1) - c(c+1)$ subject to $a + b - c = d$. The solution to this optimization problem is given in Table 3. □

Some examples of optimal tristance anticodes in $\mathcal{G}_2^{\hexagon}$ are illustrated in Fig. 8. It can be seen from Table 3 that such anticodes are regular hexagons if and only if $d \equiv 0 \pmod 3$.

Remark. Using similar methods, we can also characterize the optimal *distance* anticodes in the hexagonal graph. For even diameter d, such anticodes are regular hexagons (spheres in $\mathcal{G}_2^{\hexagon}$) centered about a lattice point $v_0 = (x_0, y_0)$, namely

$$\mathscr{S}_{d/2}(v_0) = \left\{(x, y) \in A_2 : |x - x_0| \leqslant \frac{d}{2}, \quad |y - y_0| \leqslant \frac{d}{2}, \quad |(x-y) - (x_0-y_0)| \leqslant \frac{d}{2}\right\}.$$

For odd d, optimal distance anticodes are again "spheres" of radius $(d+1)/2$, but no longer centered about a lattice point. Specifically, an optimal distance anticode of an odd diameter d is given by

$$\left\{(x,y)\in A_2 : |x-x_0'|\leqslant\frac{d+1}{2},\ |y-y_0'|\leqslant\frac{d+1}{2},\ |(x-y)-(x_0'-y_0')|\leqslant\frac{d+1}{2}\right\},$$

where $(x_0', y_0') = v_0 + \xi$, with $v_0 = (x_0, y_0)$ being an arbitrary point of A_2 and ξ being one of $(^1/_3, -^1/_3)$, $(^2/_3, ^1/_3)$, $(^1/_3, ^2/_3)$, $(-^1/_3, ^1/_3)$, $(-^2/_3, -^1/_3)$, $(-^1/_3, -^2/_3)$. Such anticodes can be construed as regular hexagons over $\mathbb{R}^2 \simeq \mathbb{C}$, but they are not regular hexagons when viewed as subsets of A_2. The cardinality of an optimal distance anticode of diameter d is $1 + 3d(d+2)/4$ if d is even, and $3(d+1)^2/4$ if d is odd.

4. Higher dimensions and higher dispersions

In general, extending our results for tristance anticodes in $\mathbb{Z}^2$ to higher dimensions and/or higher dispersions appears to be a difficult problem. Nevertheless, we pursue such generalizations in this section, in part to illustrate the difficulties that arise along the way.

In §4.1, we study tristance anticodes in $\mathbb{Z}^3$. Here, the general approach developed in §2 still works: we first characterize the optimal centered anticodes $\mathcal{A}_d(v_1, v_2) \subset \mathbb{Z}^3$ and thus determine, using Lemma 10, the shape of an optimal unrestricted tristance anticode $\mathcal{A}_d \subseteq \mathbb{Z}^3$. The problem is that the expressions for the diameter and the cardinality of $\mathcal{A}_d$ are much more involved than their counterparts for $\mathbb{Z}^2$ in (14) and (15). The resulting optimization task involves 23 variables and does not appear to be tractable. We conjecture, however, that optimal tristance anticodes in $\mathbb{Z}^3$ satisfy a certain symmetry condition. Subject to this conjecture, we determine the parameters of such anticodes and their cardinality.

In §4.2, we consider quadristance anticodes in $\mathbb{Z}^2$. This serves to illustrate a situation where the approach of §2 breaks down. We can still characterize the optimal centered quadristance anticodes $\mathcal{A}_d(z_1, z_2, z_3) \subset \mathbb{Z}^2$ and (the appropriate generalization of) Lemma 10 still applies. However, such centered anticodes are no longer convex and their general shape is *not* preserved under intersection. Thus Lemma 10 tells us nothing about the shape of unrestricted optimal quadristance anticodes in $\mathbb{Z}^2$. We use the octagon shape to derive a lower bound on the cardinality of such anticodes. We conjecture that this bound is, in fact, exact. Observe, however, that shapes other than octagons occur among optimal quadristance anticodes, at least for certain diameters (cf. Fig. 12).

4.1. *Optimal tristance anticodes in the grid graph of $\mathbb{Z}^3$*

We first need an expression for tristance in $\mathcal{G}_3^{\boxplus}$, the grid graph of $\mathbb{Z}^3$. Fortunately, the tristance formula of Theorem 1 easily generalizes to arbitrary dimensions.

Theorem 28. *Let $v = (v_1, v_2, \ldots, v_n)$, $v' = (v_1', v_2', \ldots, v_n')$, and $v'' = (v_1'', v_2'', \ldots, v_n'')$ be distinct points in $\mathbb{Z}^n$. Then*

$$d_3(v, v', v'') = \sum_{i=1}^{n}\Bigl(\max\{v_i, v_i', v_i''\} - \min\{v_i, v_i', v_i''\}\Bigr). \tag{41}$$

Proof. It is easy to see that, for all $i = 1, 2, \ldots, n$, any spanning tree for v, v', v'' must contain at least $\max\{v_i, v'_i, v''_i\} - \min\{v_i, v'_i, v''_i\}$ edges that are parallel to the ith coordinate axis. Thus the sum on the right-hand side of (41) is a lower bound on $d_3(v, v', v'')$. To show that this bound holds with equality, we use induction on n, with Theorem 1 serving as the induction base. Assume w.l.o.g. that $v'_n \leqslant v_n \leqslant v''_n$ and let $u, w \in \mathbb{Z}^n$ be defined by

$$u = (v'_1, v'_2, \ldots, v'_{n-1}, v_n) \qquad \text{and} \qquad w = (v''_1, v''_2, \ldots, v''_{n-1}, v_n).$$

It takes $v_n - v'_n$ edges to connect v' with u and another $v''_n - v_n$ edges to connect v'' with w, altogether $v''_n - v'_n = \max\{v_n, v'_n, v''_n\} - \min\{v_n, v'_n, v''_n\}$ edges. Since the points u, v, w belong to the same coset of $\mathbb{Z}^{n-1}$ in $\mathbb{Z}^n$, the claim now follows by induction hypothesis. $\square$

Next, we generalize to three dimensions the definition of a bounding rectangle in §2.1. Let $v_1 = (x_1, y_1, z_1), v_2 = (x_2, y_2, z_2), \ldots, v_n = (x_n, y_n, z_n)$ be n distinct points in $\mathbb{Z}^3$. Then the *bounding cuboid* of $v_1, v_2, \ldots, v_n$ is the smallest cuboid $\mathcal{C}(v_1, v_2, \ldots, v_n)$ with edges parallel to the axes that contains all the n points. Explicitly, define $x_{\max} = \max\{x_1, x_2, \ldots, x_n\}$ and $x_{\min} = \min\{x_1, x_2, \ldots, x_n\}$. Let $y_{\max}$, $y_{\min}$, $z_{\max}$, and $z_{\min}$ be defined similarly. Then

$$\mathcal{C}(v_1, v_2, \ldots, v_n) \stackrel{\text{def}}{=} \left\{(x, y, z) \in \mathbb{Z}^3 : x_{\min} \leqslant x \leqslant x_{\max},\ y_{\min} \leqslant y \leqslant y_{\max},\ z_{\min} \leqslant z \leqslant z_{\max}\right\}.$$

By Theorem 28, the tristance of *any* three points that lie in the bounding cuboid $\mathcal{C}(v_1, v_2)$ of $v_1 = (x_1, y_1, z_1)$ and $v_2 = (x_2, y_2, z_2)$ is at most $d(v_1, v_2) = |x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|$. This immediately leads to the following characterization of optimal tristance anticodes in $\mathbb{Z}^3$ that are centered about two given points $v_1, v_2 \in \mathbb{Z}^3$ (cf. Theorem 8).

Theorem 29. *Let $v_1 = (x_1, y_1, z_1)$, $v_2 = (x_2, y_2, z_2)$ be distinct points in $\mathbb{Z}^3$. Let $\mathcal{A}_d(v_1, v_2)$ be the optimal tristance anticode in $\mathcal{G}_3^{\boxplus}$ of diameter $d \geqslant d(v_1, v_2)$ centered about v_1 and v_2. Write $\delta = d - d(v_1, v_2)$. Let $x_{\max} = \max\{x_1, x_2\}$, $x_{\min} = \min\{x_1, x_2\}$ with $y_{\max}, y_{\min}$ and $z_{\max}, z_{\min}$ defined similarly. Then $\mathcal{A}_d(v_1, v_2)$ consists of all $v = (x, y, z)$ in $\mathbb{Z}^3$ such that*

$$x_{\min} - \delta \leqslant x \leqslant x_{\max} + \delta, \tag{42}$$
$$y_{\min} - \delta \leqslant y \leqslant y_{\max} + \delta, \tag{43}$$
$$z_{\min} - \delta \leqslant z \leqslant z_{\max} + \delta, \tag{44}$$
$$x_{\min} + y_{\min} - \delta \leqslant x + y \leqslant x_{\max} + y_{\max} + \delta, \tag{45}$$
$$x_{\min} - y_{\max} - \delta \leqslant x - y \leqslant x_{\max} - y_{\min} + \delta, \tag{46}$$
$$x_{\min} + z_{\min} - \delta \leqslant x + z \leqslant x_{\max} + z_{\max} + \delta, \tag{47}$$
$$x_{\min} - z_{\max} - \delta \leqslant x - z \leqslant x_{\max} - z_{\min} + \delta, \tag{48}$$
$$y_{\min} + z_{\min} - \delta \leqslant y + z \leqslant y_{\max} + z_{\max} + \delta, \tag{49}$$
$$y_{\min} - z_{\max} - \delta \leqslant y - z \leqslant y_{\max} - z_{\min} + \delta, \tag{50}$$
$$x_{\min} + y_{\min} + z_{\min} - \delta \leqslant x + y + z \leqslant x_{\max} + y_{\max} + z_{\max} + \delta, \tag{51}$$
$$x_{\min} - y_{\max} + z_{\min} - \delta \leqslant x - y + z \leqslant x_{\max} - y_{\min} + z_{\max} + \delta, \tag{52}$$
$$x_{\min} + y_{\min} - z_{\max} - \delta \leqslant x + y - z \leqslant x_{\max} + y_{\max} - z_{\min} + \delta, \tag{53}$$
$$x_{\min} - y_{\max} - z_{\max} - \delta \leqslant x - y - z \leqslant x_{\max} - y_{\min} - z_{\min} + \delta. \tag{54}$$

Proof. It follows from Theorem 28 that $\mathcal{A}_d(v_1, v_2) = \mathcal{C}(v_1, v_2)$ if $\delta = 0$. Hence for $\delta > 0$, the set $\mathcal{A}_d(v_1, v_2) = \{v \in \mathbb{Z}^3 : d_3(v_1, v_2, v) \leqslant d(v_1, v_2) + \delta\}$ consists of all points

$(x, y, z) \in \mathbb{Z}^3$ whose L_1-distance from the bounding cuboid $\mathcal{C}(v_1, v_2)$ is at most δ. It can be readily verified that this is precisely the set described by Eqs. (42)–(54). $\square$

The centered anticode $\mathcal{A}_d(v_1, v_2)$ in Theorem 29 is an example of a set we call the icosihexahedron. In general, we define an *icosihexahedron* $\mathscr{I}(\alpha_1, \alpha_2, \ldots, \alpha_{26})$ as the set of all points of $\mathbb{Z}^3$ that lie within the convex polyhedron with 26 faces, given by the inequalities

$$\alpha_1 \leqslant x \leqslant \alpha_{14}, \qquad \alpha_2 \leqslant y \leqslant \alpha_{15}, \qquad \alpha_3 \leqslant z \leqslant \alpha_{16}, \tag{55}$$

$$\alpha_4 \leqslant x + y \leqslant \alpha_{17}, \qquad \alpha_5 \leqslant x + z \leqslant \alpha_{18}, \qquad \alpha_6 \leqslant y + z \leqslant \alpha_{19}, \tag{56}$$

$$\alpha_7 \leqslant x - y \leqslant \alpha_{20}, \qquad \alpha_8 \leqslant x - z \leqslant \alpha_{21}, \qquad \alpha_9 \leqslant y - z \leqslant \alpha_{22}, \tag{57}$$

$$\alpha_{10} \leqslant x + y + z \leqslant \alpha_{23}, \qquad \alpha_{11} \leqslant x - y - z \leqslant \alpha_{24}, \tag{58}$$

$$\alpha_{12} \leqslant x - y + z \leqslant \alpha_{25}, \qquad \alpha_{13} \leqslant x + y - z \leqslant \alpha_{26}. \tag{59}$$

It is clear from (55)–(59) that an intersection of two icosihexahedra is again an icosihexahedron. Along with Lemma 10 and Theorem 29, this immediately implies the following:

Corollary 30. *Let $\mathcal{A}_d$ be an optimal tristance anticode of diameter d in $\mathcal{G}_3^{\boxplus}$. Then $\mathcal{A}_d$ is an icosihexahedron $\mathscr{I}(\alpha_1, \alpha_2, \ldots, \alpha_{26})$ for some $\alpha_1, \alpha_2, \ldots, \alpha_{26} \in \mathbb{Z}$.*

As in §2.3, we can assume that $\alpha_1 = \alpha_2 = \alpha_3 = 0$ in (55), up to a translation in $\mathbb{Z}^3$. We furthermore re-parametrize an icosihexahedron $\mathscr{I}(0, 0, 0, \alpha_4, \alpha_5, \ldots, \alpha_{26})$ as follows:

$$0 \leqslant x \leqslant a, \qquad 0 \leqslant y \leqslant b, \qquad 0 \leqslant z \leqslant c, \tag{60}$$

$$e_{\bar{x}\bar{y}} \leqslant x + y \leqslant a + b - e_{xy}, \tag{61}$$

$$e_{\bar{x}y} - b \leqslant x - y \leqslant a - e_{x\bar{y}}, \tag{62}$$

$$e_{\bar{x}\bar{z}} \leqslant x + z \leqslant a + c - e_{xz}, \tag{63}$$

$$e_{\bar{x}z} - c \leqslant x - z \leqslant a - e_{x\bar{z}}, \tag{64}$$

$$e_{\bar{y}\bar{z}} \leqslant y + z \leqslant b + c - e_{yz}, \tag{65}$$

$$e_{\bar{y}z} - c \leqslant y - z \leqslant b - e_{y\bar{z}}, \tag{66}$$

$$\theta_{\bar{x}\bar{y}\bar{z}} \leqslant x + y + z \leqslant a + b + c - \theta_{xyz}, \tag{67}$$

$$\theta_{\bar{x}y\bar{z}} - b \leqslant x - y + z \leqslant a + c - \theta_{x\bar{y}z}, \tag{68}$$

$$\theta_{\bar{x}\bar{y}z} - c \leqslant x + y - z \leqslant a + b - \theta_{xy\bar{z}}, \tag{69}$$

$$\theta_{\bar{x}yz} - b - c \leqslant x - y - z \leqslant a - \theta_{x\bar{y}\bar{z}}, \tag{70}$$

where $a = \alpha_{14}$, $b = \alpha_{15}$, $c = \alpha_{16}$ while $e_{\bar{x}\bar{y}} = \alpha_4$, $e_{xy} = \alpha_{14} + \alpha_{15} - \alpha_{17}$, $e_{\bar{x}y} = \alpha_7 + \alpha_{15}$, $e_{x\bar{y}} = \alpha_{14} - \alpha_{20}$, $e_{\bar{x}\bar{z}} = \alpha_5$, $e_{xz} = \alpha_{14} + \alpha_{16} - \alpha_{18}$, $e_{\bar{x}z} = \alpha_8 + \alpha_{16}$, $e_{x\bar{z}} = \alpha_{14} - \alpha_{21}$, $e_{\bar{y}\bar{z}} = \alpha_6$, $e_{yz} = \alpha_{15} + \alpha_{16} - \alpha_{19}$, $e_{\bar{y}z} = \alpha_9 + \alpha_{16}$, $e_{y\bar{z}} = \alpha_{15} - \alpha_{22}$, and $\theta_{\bar{x}\bar{y}\bar{z}} = \alpha_{10}$, $\theta_{\bar{x}\bar{y}z} = \alpha_{13} + \alpha_{16}$, $\theta_{\bar{x}y\bar{z}} = \alpha_{12} + \alpha_{15}$, $\theta_{\bar{x}yz} = \alpha_{11} + \alpha_{15} + \alpha_{16}$, $\theta_{x\bar{y}\bar{z}} = \alpha_{14} - \alpha_{24}$, $\theta_{x\bar{y}z} = \alpha_{14} + \alpha_{16} - \alpha_{25}$, $\theta_{xy\bar{z}} = \alpha_{14} + \alpha_{15} - \alpha_{26}$, $\theta_{xyz} = \alpha_{14} + \alpha_{15} + \alpha_{16} - \alpha_{23}$.

Eqs. (60)–(70) make it apparent that an icosihexahedron is just a truncated cuboid: the eight values $\theta_{\bar{x}\bar{y}\bar{z}}, \theta_{\bar{x}\bar{y}z}, \ldots, \theta_{xyz}$ give the amount of truncation at the vertices, while the twelve values $e_{\bar{x}\bar{y}}, e_{\bar{x}y}, \ldots, e_{yz}$ describe the amount of truncation along the edges. Fig. 9 shows a generic icosihexahedron as a 26-faceted three-dimensional solid along with our labeling of the edges and vertices of the corresponding cuboid, as reflected in (60)–(70).

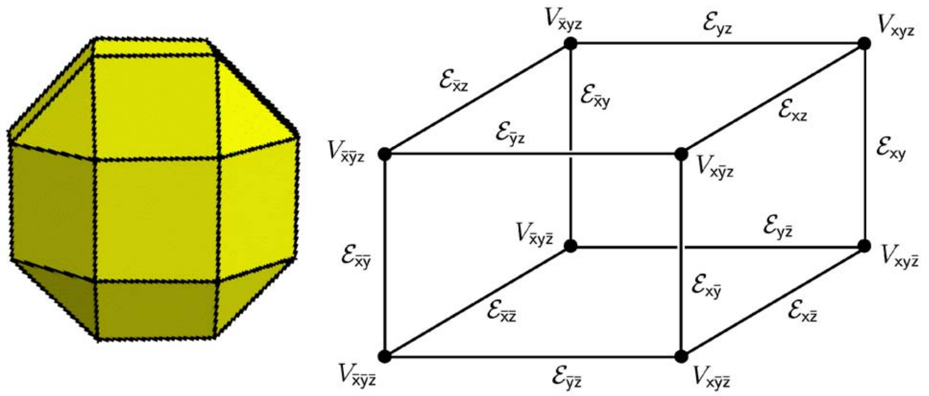

Fig. 9. A generic icosihexahedron and a labeling of its 20 truncations.

Observe that one can assume w.l.o.g. that each of the 26 inequalities in (55)–(59) holds with equality for at least one point of $\mathscr{I}(\alpha_1, \alpha_2, \ldots, \alpha_{26})$—otherwise, we can always increase the corresponding constant α_i if $i \leqslant 13$ or decrease it if $i \geqslant 14$. This implies that each of the inequalities in (60)–(70) must also hold with equality for at least one point of the icosihexahedron. We will make use of this observation later on. We next determine the diameter of an icosihexahedron $\mathscr{I}(a, b, c, \{e_{\chi\psi}, e_{\chi\omega}, e_{\psi\omega}\}, \{\theta_{\chi\psi\omega}\})$ parametrized as in (60)–(70). To this end, we need to consider certain configurations of edges and vertices of a cuboid. Referring to Fig. 9, we define $\bar{\bar{\mathsf{x}}} = \mathsf{x}$, $\bar{\bar{\mathsf{y}}} = \mathsf{y}$, and $\bar{\bar{\mathsf{z}}} = \mathsf{z}$, so that the complement operation $\bar{\cdot}$ is an involution, as expected. Let χ denote x or $\bar{\mathsf{x}}$, let ψ denote y or $\bar{\mathsf{y}}$, and let ω denote z or $\bar{\mathsf{z}}$. With this notation, we say that a vertex $V_{\chi\psi\omega}$ *lies opposite* the edges $\mathcal{E}_{\bar{\chi}\bar{\psi}}$, $\mathcal{E}_{\bar{\chi}\bar{\omega}}$, and $\mathcal{E}_{\bar{\psi}\bar{\omega}}$ (indeed, these are the three edges incident upon the diagonally opposite vertex $V_{\bar{\chi}\bar{\psi}\bar{\omega}}$). We also say that the edges $\mathcal{E}_{\chi\psi}$, $\mathcal{E}_{\bar{\chi}\omega}$, $\mathcal{E}_{\bar{\psi}\bar{\omega}}$ *span the cuboid* (these are the 8 possible choices of three edges such that each face contains one of them).

Lemma 31. *Let $\mathscr{I}(a, b, c, \{e_{\chi\psi}, e_{\chi\omega}, e_{\psi\omega}\}, \{\theta_{\chi\psi\omega}\})$ be an icosihexahedron, parametrized as in* (60)–(70). *Define*

$$s \stackrel{\text{def}}{=} \min_{\chi,\psi,\omega} \left\{ e_{\chi\psi} + e_{\bar{\chi}\omega} + e_{\bar{\psi}\bar{\omega}} \right\}, \tag{71}$$

$$t \stackrel{\text{def}}{=} \min_{\chi,\psi,\omega} \left\{ \min\{e_{\bar{\chi}\bar{\psi}}, e_{\bar{\chi}\bar{\omega}}, e_{\bar{\psi}\bar{\omega}}\} + \theta_{\chi\psi\omega} \right\}, \tag{72}$$

where the minimum in (71) *and* (72) *is taken over the eight possible assignments of values to χ, ψ, and ω. Then the diameter of $\mathscr{I}(a, b, c, \{e_{\chi\psi}, e_{\chi\omega}, e_{\psi\omega}\}, \{\theta_{\chi\psi\omega}\})$ is given by*

$$d = a + b + c - \min\{s, t\}. \tag{73}$$

Proof. Let $v_1 = (x_1, y_1, z_1)$, $v_2 = (x_2, y_2, z_2)$, and $v_3 = (x_3, y_3, z_3)$ be a triple of distinct points of $\mathscr{I}(a, b, c, \{e_{\chi\psi}, e_{\chi\omega}, e_{\psi\omega}\}, \{\theta_{\chi\psi\omega}\})$ and consider their bounding cuboid

$\mathcal{C}(v_1, v_2, v_3)$. The key observation is that each of the six faces of $\mathcal{C}(v_1, v_2, v_3)$ must contain at least one of the three points. This leads to the following three cases.

Case 1: Suppose that none of the points v_1, v_2, v_3 is a vertex of $\mathcal{C}(v_1, v_2, v_3)$. Then each of the points v_1, v_2, v_3 must belong to an edge of $\mathcal{C}(v_1, v_2, v_3)$ and, moreover, the three edges must span the cuboid. Thus if the cuboid $\mathcal{C}(v_1, v_2, v_3)$ is labeled as in Fig. 9, we can assume w.l.o.g. that $v_1 \in \mathcal{E}_{\chi\psi}$, $v_2 \in \mathcal{E}_{\bar{\chi}\omega}$, and $v_3 \in \mathcal{E}_{\bar{\psi}\bar{\omega}}$ for some χ, ψ, ω. Referring to Fig. 9, it follows that in the definition of $\mathcal{C}(v_1, v_2, v_3)$, we must have $x_{\max} - x_{\min} = |x_1 - x_2|$, $y_{\max} - y_{\min} = |y_1 - y_3|$, and $z_{\max} - z_{\min} = |z_2 - z_3|$. Thus

$$\begin{aligned} d_3(v_1, v_2, v_3) &= |x_1 - x_2| + |y_1 - y_3| + |z_2 - z_3| \\ &= \pm(x_1 \pm y_1) \mp (x_2 \pm z_2) \mp (y_3 \mp z_3). \end{aligned}$$

There are eight cases, depending on the 8 possible values of χ, ψ, ω; but in each case $x_1 \pm y_1$ is bounded by (61) or (62), $x_2 \pm z_2$ is bounded by (63) or (64), and $y_3 \mp z_3$ is bounded by (65) or (66). In all cases, these bounds produce the same result, namely

$$d_3(v_1, v_2, v_3) \leqslant a + b + c - \left(e_{\chi\psi} + e_{\bar{\chi}\omega} + e_{\bar{\psi}\bar{\omega}}\right) \leqslant a + b + c - s. \tag{74}$$

Note that we can always achieve the second inequality in (74) with equality by choosing $v_1 \in \mathcal{E}_{\chi\psi}$, $v_2 \in \mathcal{E}_{\bar{\chi}\omega}$, $v_3 \in \mathcal{E}_{\bar{\psi}\bar{\omega}}$ so that χ, ψ, ω attains the minimum in (71). The first inequality in (74) can be also achieved with equality, because of the assumption that each of (61)–(66) holds with equality for some point of the icosihexahedron.

Case 2: Suppose that one of the three points v_1, v_2, v_3 is a vertex of $\mathcal{C}(v_1, v_2, v_3)$, say $v_1 = V_{\chi\psi\omega}$, but none of the other two points is in the diagonally opposite vertex $V_{\bar{\chi}\bar{\psi}\bar{\omega}}$. Then one of v_2, v_3 must belong to an edge that lies opposite $V_{\chi\psi\omega}$, say $v_2 \in \mathcal{E}_{\bar{\chi}\bar{\psi}}$, while the other point must belong to the remaining face of $\mathcal{C}(v_1, v_2, v_3)$. Referring once again to Fig. 9, we see that $x_{\max} - x_{\min} = |x_1 - x_2|$, $y_{\max} - y_{\min} = |y_1 - y_2|$, and $z_{\max} - z_{\min} = |z_1 - z_3|$. It follows that

$$\begin{aligned} d_3(v_1, v_2, v_3) &= |x_1 - x_2| + |y_1 - y_2| + |z_1 - z_3| \\ &= \pm(x_1 \pm y_1 \pm z_1) \mp (x_2 \pm y_2) \mp z_3. \end{aligned}$$

As before, there are eight cases depending on the values of χ, ψ, ω, but in each case $x_1 \pm y_1 \pm z_1$ is bounded by one of (67)–(70), $x_2 \pm y_2$ is bounded by (61) or (62), and $\mp z_3$ is bounded by (60). In all the eight cases, we get the same result, namely

$$d_3(v_1, v_2, v_3) \leqslant a + b + c - \left(e_{\bar{\chi}\bar{\psi}} + \theta_{\chi\psi\omega}\right) \leqslant a + b + c - t. \tag{75}$$

Once again, we can attain the first inequality in (75) with equality by choosing suitable points v_1, v_2, v_3 in the icosihexahedron. The other two cases where $v_2 \in \mathcal{E}_{\bar{\chi}\bar{\omega}}$ or $v_2 \in \mathcal{E}_{\bar{\psi}\bar{\omega}}$ are similar, leading to the minimization among $e_{\bar{\chi}\bar{\psi}}, e_{\bar{\chi}\bar{\omega}}, e_{\bar{\psi}\bar{\omega}}$ in (72).

Case 3: Now suppose that one of the points v_1, v_2, v_3 is a vertex of $\mathcal{C}(v_1, v_2, v_3)$ and another of the points is the diagonally opposite vertex, say $v_1 = V_{\chi\psi\omega}$ and $v_2 = V_{\bar{\chi}\bar{\psi}\bar{\omega}}$. Then $x_{\max} - x_{\min} = |x_1 - x_2|$, $y_{\max} - y_{\min} = |y_1 - y_2|$, and $z_{\max} - z_{\min} = |z_1 - z_2|$, so

$$\begin{aligned} d_3(v_1, v_2, v_3) &= |x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2| \\ &= \pm(x_1 \pm y_1 \pm z_1) \mp (x_2 \pm y_2 \pm z_2). \end{aligned}$$

We again get eight cases depending on the values of χ, ψ, ω, with $x_1 \pm y_1 \pm z_1$ and $x_2 \pm y_2 \pm z_2$ both bounded by the same equation—one of (67)–(70)—one from above and the other from below. This produces

$$d_3(v_1, v_2, v_3) \leqslant a + b + c - \left(\theta_{\chi\psi\omega} + \theta_{\bar{\chi}\bar{\psi}\bar{\omega}}\right). \tag{76}$$

We can again achieve the bound in (76) with equality but, as we shall see, this case does not produce a diametric triple of points in $\mathscr{I}(a, b, c, \{e_{\chi\psi}, e_{\chi\omega}, e_{\psi\omega}\}, \{\theta_{\chi\psi\omega}\})$.

Since the three cases above are exhaustive, in order to complete the proof of the lemma, it would suffice to show that

$$\theta_{\bar{\chi}\bar{\psi}\bar{\omega}} \geqslant \frac{e_{\bar{\chi}\bar{\psi}} + e_{\bar{\chi}\bar{\omega}} + e_{\bar{\psi}\bar{\omega}}}{2} \geqslant \min\left\{e_{\bar{\chi}\bar{\psi}}, e_{\bar{\chi}\bar{\omega}}, e_{\bar{\psi}\bar{\omega}}\right\} \tag{77}$$

which, in conjunction with (76), would imply that $d_3(v_1, v_2, v_3) \leqslant a + b + c - t$ in Case 3. This follows from the fact that each of the 26 inequalities in (60)–(70) must hold with equality at some point of the icosihexahedron. For example, let $\chi = \mathsf{x}$, $\psi = \mathsf{y}$, and $\omega = \mathsf{z}$. Adding the first inequalities of (61), (63), (65) yields $2(x + y + z) \geqslant e_{\bar{\mathsf{x}}\bar{\mathsf{y}}} + e_{\bar{\mathsf{x}}\bar{\mathsf{z}}} + e_{\bar{\mathsf{y}}\bar{\mathsf{z}}}$. Thus if the first inequality in (67) is to hold with equality, we must have $\theta_{\bar{\mathsf{x}}\bar{\mathsf{y}}\bar{\mathsf{z}}} \geqslant 1/2\left(e_{\bar{\mathsf{x}}\bar{\mathsf{y}}} + e_{\bar{\mathsf{x}}\bar{\mathsf{z}}} + e_{\bar{\mathsf{y}}\bar{\mathsf{z}}}\right)$. The other seven ways to assign values to χ, ψ, ω can be treated similarly. $\square$

The next task is to determine the volume of $\mathscr{I}(a, b, c, \{e_{\chi\psi}, e_{\chi\omega}, e_{\psi\omega}\}, \{\theta_{\chi\psi\omega}\})$ in terms of its parameters. This innocuous task is surprisingly arduous: a complete expression for $|\mathscr{I}(a, b, c, \{e_{\chi\psi}, e_{\chi\omega}, e_{\psi\omega}\}, \{\theta_{\chi\psi\omega}\})|$ would entail hundreds of cases depending upon the relationships between various parameters. Moreover, given such an expression, we would need to solve a nonlinear integer optimization problem involving 23 variables—the parameters $a, b, c, \{e_{\chi\psi}, e_{\chi\omega}, e_{\psi\omega}\}, \{\theta_{\chi\psi\omega}\}$ in (60)–(70). This problem does not appear to be tractable. The situation simplifies considerably, however, with the help of the following:

Conjecture 32. *For each diameter $d \geqslant 2$, there exists an optimal tristance anticode in $\mathcal{G}_3^{\boxplus}$ $\mathcal{A}_d = \mathscr{I}(a, b, c, \{e_{\chi\psi}, e_{\chi\omega}, e_{\psi\omega}\}, \{\theta_{\chi\psi\omega}\})$ with equally truncated edges*; *that is, such that*

$$e_{\bar{\mathsf{x}}\bar{\mathsf{y}}} = e_{\bar{\mathsf{x}}\mathsf{y}} = e_{\bar{\mathsf{x}}\bar{\mathsf{z}}} = e_{\bar{\mathsf{x}}\mathsf{z}} = e_{\bar{\mathsf{y}}\bar{\mathsf{z}}} = e_{\bar{\mathsf{y}}\bar{\mathsf{z}}} = e_{\mathsf{y}\bar{\mathsf{z}}} = e_{\mathsf{y}\mathsf{z}} = e_{\mathsf{x}\bar{\mathsf{z}}} = e_{\mathsf{x}\mathsf{z}} = e_{\mathsf{x}\bar{\mathsf{y}}} = e_{\mathsf{x}\mathsf{y}} \stackrel{\text{def}}{=} e. \tag{78}$$

It is easy to see from (72) that if $\mathcal{A}_d = \mathscr{I}(a, b, c, \{e_{\chi\psi}, e_{\chi\omega}, e_{\psi\omega}\}, \{\theta_{\chi\psi\omega}\})$ is an optimal anticode with equally truncated edges, then its vertices must also be equally truncated:

$$\theta_{\bar{\mathsf{x}}\bar{\mathsf{y}}\bar{\mathsf{z}}} = \theta_{\bar{\mathsf{x}}\bar{\mathsf{y}}\mathsf{z}} = \theta_{\bar{\mathsf{x}}\mathsf{y}\bar{\mathsf{z}}} = \theta_{\bar{\mathsf{x}}\mathsf{y}\mathsf{z}} = \theta_{\mathsf{x}\bar{\mathsf{y}}\bar{\mathsf{z}}} = \theta_{\mathsf{x}\bar{\mathsf{y}}\mathsf{z}} = \theta_{\mathsf{x}\mathsf{y}\bar{\mathsf{z}}} = \theta_{\mathsf{x}\mathsf{y}\mathsf{z}} \stackrel{\text{def}}{=} \theta. \tag{79}$$

We will denote an icosihexahedron satisfying (78) and (79) as $\mathscr{I}(a, b, c, e, \theta)$. It now follows from (71)–(73) and (77) that if $\mathcal{A}_d = \mathscr{I}(a, b, c, e, \theta)$ then

$$\frac{3e}{2} \leqslant \theta \leqslant 2e. \tag{80}$$

The condition that each of the inequalities in (60)–(70) holds with equality at some point of $\mathscr{I}(a, b, c, e, \theta)$ further implies that $2e \leqslant \min\{a, b, c\}$. We present the next lemma without proof; while its proof is not conceptually difficult, it is rather tedious.

Table 4
Parameters of (conjecturally) optimal tristance anticodes in $\mathcal{G}_3^{\boxplus}$

d (mod 3)	a	b	c	e
0	$\frac{d}{3}+e$	$\frac{d}{3}+e$	$\frac{d}{3}+e$	$(d+1)-\left[\sqrt{{}^2\!/_3\,(d+1)(d+2)}\right]$
1	$\frac{d+2}{3}+e$	$\frac{d-1}{3}+e$	$\frac{d-1}{3}+e$	$(d+1)-\left[\sqrt{{}^2\!/_3\,(d+1)(d+2)+{}^1\!/_3}\right]$
2	$\frac{d+1}{3}+e$	$\frac{d+1}{3}+e$	$\frac{d-2}{3}+e$	$(d+1)-\left[\sqrt{{}^2\!/_3\,(d+1)(d+2)+{}^1\!/_3}\right]$

Lemma 33. *Subject to the condition* ${}^3\!/_2\, e \leqslant \theta \leqslant 2e \leqslant \min\{a, b, c\}$, *the volume of an* icosihexahedron $\mathscr{I}(a,b,c,e,\theta)$ *is given by*

$$|\mathscr{I}(a,b,c,e,\theta)| = (a+1)(b+1)(c+1) \;-\; 2e(e+1)(a+b+c+3) \\ +24e^3 \;+\; \frac{4}{3}\,\theta\Big(3\theta(6e-1) - 9e(3e-1) - (2\theta+1)(2\theta-1)\Big).$$

Using the expression for $|\mathscr{I}(a,b,c,e,\theta)|$ in Lemma 33 along with (73), it can be furthermore shown that if $A_d = \mathscr{I}(a,b,c,e,\theta)$ is an optimal tristance anticode in $\mathcal{G}_3^{\boxplus}$, then $\theta = 2e$. With this, the expression for the volume of the icosihexahedron further simplifies to

$$|\mathscr{I}(a,b,c,e,2e)| = (a+1)(b+1)(c+1) \\ -\,2e(e+1)\left(a+b+c+3-\frac{4}{3}\,(2e+1)\right). \tag{81}$$

It remains to maximize the cubic on the right-hand side of (81) subject to the constraints $a+b+c-3e = d$ and $a \geqslant b \geqslant c \geqslant 2e$. Note that for each fixed e, we have

$$|\mathscr{I}(a,b,c,e,2e)| \;=\; (a+1)(b+1)(c+1) \;-\; \text{const}$$

since $a+b+c = d+3e$. This immediately shows that the optimal values of a, b, c are given by $c = \lfloor d/3 \rfloor + e$ with a, b being equal to either c or $c+1$. The complete solution to the optimization problem is given in Table 4, where $[\mu]$ denotes the integer that is closest to the real number μ (rounding). We have verified by exhaustive computer search that the anticodes in Table 4 are, in fact, the *unique optimal tristance anticodes in* $\mathcal{G}_3^{\boxplus}$ up to diameter $d = 11$. Fig. 10 shows some of these anticodes, for diameters $d = 9, 10, 11$.

4.2. *Optimal quadristance anticodes in the grid graph of* $\mathbb{Z}^2$

Recall that, given distinct points z_1, z_2, z_3, z_4 in $\mathbb{Z}^2$, the *quadristance* $d_4(z_1, z_2, z_3, z_4)$ is defined as the number of edges in a minimal spanning tree for z_1, z_2, z_3, z_4 in the grid graph $\mathcal{G}_2^{\boxplus}$ of $\mathbb{Z}^2$. The following expression for quadristance is implicit in [9].

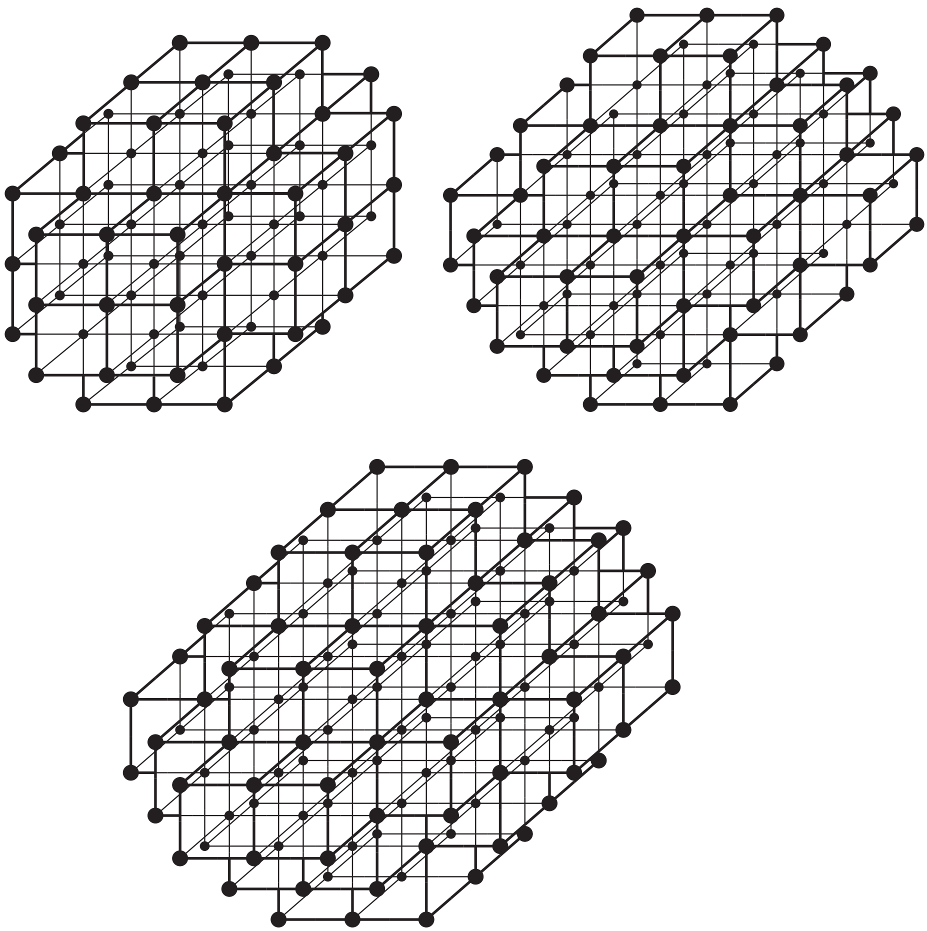

Fig. 10. Optimal tristance anticodes in $\mathcal{G}_3^{\boxplus}$ of diameter $d = 9, 10, 11$.

Theorem 34. *Let* $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2)$, $z_3 = (x_3, y_3)$, $z_4 = (x_4, y_4)$ *be distinct points in* $\mathbb{Z}^2$. *Let* σ *and* τ *be permutations of* $\{1, 2, 3, 4\}$ *such that* $x_{\sigma(1)} \leqslant x_{\sigma(2)} \leqslant x_{\sigma(3)} \leqslant x_{\sigma(4)}$ *and* $y_{\tau(1)} \leqslant y_{\tau(2)} \leqslant y_{\tau(3)} \leqslant y_{\tau(4)}$. *Then*

$$d_4(z_1, z_2, z_3, z_4) = \Big(x_{\sigma(4)} - x_{\sigma(1)}\Big) + \Big(y_{\tau(4)} - y_{\tau(1)}\Big) \tag{82}$$

provided $\tau\sigma^{-1} \in \Gamma$, *where* Γ *is the subgroup of the symmetric group generated by the permutations* $(1, 2)$, $(3, 4)$, *and* $(1, 3)(2, 4)$. *If* $\tau\sigma^{-1} \notin \Gamma$ *then*

$$\begin{aligned} d_4(z_1, z_2, z_3, z_4) = {} & \Big(x_{\sigma(4)} - x_{\sigma(1)}\Big) + \Big(y_{\tau(4)} - y_{\tau(1)}\Big) \\ & + \min\Big\{x_{\sigma(3)} - x_{\sigma(2)},\, y_{\tau(3)} - y_{\tau(2)}\Big\}. \end{aligned} \tag{83}$$

Note that one can assume w.l.o.g. that σ is the identity permutation. Then $d_4(z_1, z_2, z_3, z_4)$ is given by (82) precisely in the eight cases where

$$y_1 \leqslant y_2 \leqslant y_3 \leqslant y_4, \quad y_1 \leqslant y_2 \leqslant y_4 \leqslant y_3, \quad y_2 \leqslant y_1 \leqslant y_3 \leqslant y_4, \quad y_2 \leqslant y_1 \leqslant y_4 \leqslant y_3,$$
$$y_3 \leqslant y_4 \leqslant y_2 \leqslant y_1, \quad y_3 \leqslant y_4 \leqslant y_1 \leqslant y_2, \quad y_4 \leqslant y_3 \leqslant y_1 \leqslant y_2, \quad y_4 \leqslant y_3 \leqslant y_2 \leqslant y_1.$$

We next determine the optimal quadristance anticode $\mathcal{A}_d(z_1, z_2, z_3) \subset \mathbb{Z}^2$ centered about three given points z_1, z_2, z_3, namely the set

$$\mathcal{A}_d(z_1, z_2, z_3) \stackrel{\text{def}}{=} \left\{ z \in \mathbb{Z}^2 \;:\; d_4(z_1, z_2, z_3, z) \leqslant d \right\}. \tag{84}$$

Clearly $\mathcal{A}_d(z_1, z_2, z_3) = \varnothing$ for $d < d_3(z_1, z_2, z_3)$. As before, we first consider the case where $d = d_3(z_1, z_2, z_3)$. Recall that $\mathcal{R}(z_1, z_2)$ denotes the bounding rectangle of z_1, z_2.

Lemma 35. *Let $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2)$, $z_3 = (x_3, y_3)$ be distinct points in $\mathbb{Z}^2$, and let $\Delta = d_3(z_1, z_2, z_3)$. Write $\mathcal{R}_1 = \mathcal{R}(z_1, z_2)$, $\mathcal{R}_2 = \mathcal{R}(z_1, z_3)$, and $\mathcal{R}_3 = \mathcal{R}(z_2, z_3)$. Then*

$$\mathcal{A}_\Delta(z_1, z_2, z_3) = \left(\mathcal{R}_1 \cap \mathcal{R}_2\right) \cup \left(\mathcal{R}_1 \cap \mathcal{R}_3\right) \cup \left(\mathcal{R}_2 \cap \mathcal{R}_3\right). \tag{85}$$

Proof. By definition, $z \in \mathcal{A}_\Delta(z_1, z_2, z_3)$ iff $d_4(z_1, z_2, z_3, z) = d_3(z_1, z_2, z_3)$. This happens if and only if z belongs to the vertex set of a minimal spanning tree for z_1, z_2, z_3. Hence

$$\mathcal{A}_\Delta(z_1, z_2, z_3) = \bigcup_{T(z_1, z_2, z_3)} \left\{\text{vertex set of } T(z_1, z_2, z_3)\right\},$$

where the union is over all the minimal spanning trees for z_1, z_2, z_3. Observe that given any $u, v \in \mathbb{Z}^2$, the union of all the shortest paths (minimal spanning trees) between u and v in $\mathcal{G}_2^{\boxplus}$ is precisely the bounding rectangle $\mathcal{R}(u, v)$. Now consider $\mathcal{R}(z_1, z_2, z_3)$, the bounding rectangle of z_1, z_2, z_3. Since each of the four edges of $\mathcal{R}(z_1, z_2, z_3)$ must contain at least one of the three points, at least one of z_1, z_2, z_3 must be a vertex of $\mathcal{R}(z_1, z_2, z_3)$. Thus we can assume w.l.o.g. that z_1 is a vertex of $\mathcal{R}(z_1, z_2, z_3)$. We distinguish between two cases.

Case 1: Suppose that one of the other two points, say z_3, is the opposite vertex of the bounding rectangle $\mathcal{R}(z_1, z_2, z_3)$, as illustrated on the right-hand side of Fig. 11. It is easy to see that, in this case, any minimal spanning tree for z_1, z_2, z_3 is a union of a shortest path from z_1 to z_2 with a shortest path from z_2 to z_3. Hence

$$\mathcal{A}_\Delta(z_1, z_2, z_3) = \mathcal{R}(z_1, z_2) \cup \mathcal{R}(z_2, z_3). \tag{86}$$

Since $\mathcal{R}_2 = \mathcal{R}(z_1, z_3) = \mathcal{R}(z_1, z_2, z_3)$ in this case, we have $\mathcal{R}_1 \cap \mathcal{R}_2 = \mathcal{R}(z_1, z_2)$ and $\mathcal{R}_2 \cap \mathcal{R}_3 = \mathcal{R}(z_2, z_3)$. It follows that (86) coincides with (85).

Case 2: Suppose that none of the other two points is the vertex of $\mathcal{R}(z_1, z_2, z_3)$ opposite to z_1. Then z_2 and z_3 must belong to the two edges of $\mathcal{R}(z_1, z_2, z_3)$ that lie opposite z_1. This case is illustrated on the left-hand side of Fig. 11. As in (30), let x_{mid} and y_{mid} denote the middle values among x_1, x_2, x_3 and y_1, y_2, y_3, respectively. Write $v = (x_{\text{mid}}, y_{\text{mid}})$. Then any minimal spanning tree for z_1, z_2, z_3 consists of a shortest path from z_1 to v along with the unique shortest path from v to z_2 and the unique shortest path from v to z_3. Thus

$$\mathcal{A}_\Delta(z_1, z_2, z_3) = \mathcal{R}(z_1, v) \cup \mathcal{R}(z_2, v) \cup \mathcal{R}(z_3, v). \tag{87}$$

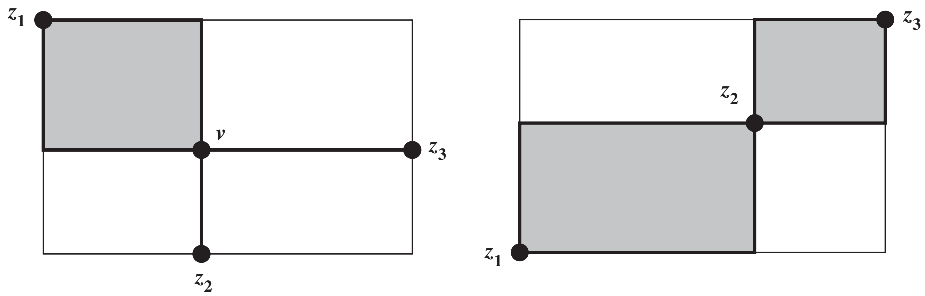

Fig. 11. Centered quadristance anticodes $\mathcal{A}_d(z_1, z_2, z_3)$ for $d = d_3(z_1, z_2, z_3)$.

Notice that $\mathcal{R}(z_1, v) = \mathcal{R}_1 \cap \mathcal{R}_2$, $\mathcal{R}(z_2, v) = \mathcal{R}_1 \cap \mathcal{R}_3$, and $\mathcal{R}(z_3, v) = \mathcal{R}_2 \cap \mathcal{R}_3$ (even though the two rectangles $\mathcal{R}(z_2, v)$, $\mathcal{R}(z_3, v)$ are degenerate). Hence, (87) again coincides with (85), and we are done. $\square$

Theorem 36. *Let $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2)$, $z_3 = (x_3, y_3)$ be distinct points in $\mathbb{Z}^2$. Let d be an integer such that $d \geqslant d_3(z_1, z_2, z_3)$, and write $c = d - d_3(z_1, z_2, z_3)$. Further define*

$$\begin{aligned}
\alpha_1 &= \max\bigl\{\min\{x_1, x_2\}, \min\{x_1, x_3\}\bigr\}, & \beta_1 &= \min\bigl\{\max\{x_1, x_2\}, \max\{x_1, x_3\}\bigr\},\\
\gamma_1 &= \max\bigl\{\min\{y_1, y_2\}, \min\{y_1, y_3\}\bigr\}, & \delta_1 &= \min\bigl\{\max\{y_1, y_2\}, \max\{y_1, y_3\}\bigr\},\\
\alpha_2 &= \max\bigl\{\min\{x_1, x_2\}, \min\{x_2, x_3\}\bigr\}, & \beta_2 &= \min\bigl\{\max\{x_1, x_2\}, \max\{x_2, x_3\}\bigr\},\\
\gamma_2 &= \max\bigl\{\min\{y_1, y_2\}, \min\{y_2, y_3\}\bigr\}, & \delta_2 &= \min\bigl\{\max\{y_1, y_2\}, \max\{y_2, y_3\}\bigr\},\\
\alpha_3 &= \max\bigl\{\min\{x_1, x_3\}, \min\{x_2, x_3\}\bigr\}, & \beta_3 &= \min\bigl\{\max\{x_1, x_3\}, \max\{x_2, x_3\}\bigr\},\\
\gamma_3 &= \max\bigl\{\min\{y_1, y_3\}, \min\{y_2, y_3\}\bigr\}, & \delta_3 &= \min\bigl\{\max\{y_1, y_3\}, \max\{y_2, y_3\}\bigr\}.
\end{aligned}$$

Then the centered quadristance anticode $\mathcal{A}_d(z_1, z_2, z_3)$ is a union of three octagons $\mathcal{O}_1$, $\mathcal{O}_2$, and $\mathcal{O}_3$, where for $i = 1, 2, 3$, the octagon $\mathcal{O}_i$ consists of all $(x, y) \in \mathbb{Z}^2$ such that

$$\alpha_i - c \leqslant x \leqslant \beta_i + c, \quad \alpha_i + \gamma_i - c \leqslant x + y \leqslant \beta_i + \delta_i + c,$$

$$\gamma_i - c \leqslant y \leqslant \delta_i + c, \quad \alpha_i - \delta_i - c \leqslant x - y \leqslant \beta_i - \gamma_i + c.$$

Proof. As in Lemma 35, let $\Delta = d_3(z_1, z_2, z_3)$. It is not difficult to show that $z \in \mathcal{A}_d(z_1, z_2, z_3)$ if and only if the L_1-distance from z to (the closest point of) $\mathcal{A}_\Delta(z_1, z_2, z_3)$ is at most $c = d - \Delta$. Lemma 35 proves that $\mathcal{A}_\Delta(z_1, z_2, z_3)$ is a union of three rectangles. For each rectangle, the set of all $z \in \mathbb{Z}^2$ that are at L_1-distance at most c from it is an octagon. Indeed, $\mathcal{O}_1$ is precisely the set of all points that are at L_1-distance at most c from $\mathcal{R}_1 \cap \mathcal{R}_2$, while $\mathcal{O}_2$ and $\mathcal{O}_3$ are constructed similarly with respect to $\mathcal{R}_1 \cap \mathcal{R}_3$ and $\mathcal{R}_2 \cap \mathcal{R}_3$, where $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3$ are as defined in Lemma 35. Hence $\mathcal{A}_d(z_1, z_2, z_3) = \mathcal{O}_1 \cup \mathcal{O}_2 \cup \mathcal{O}_3$. $\square$

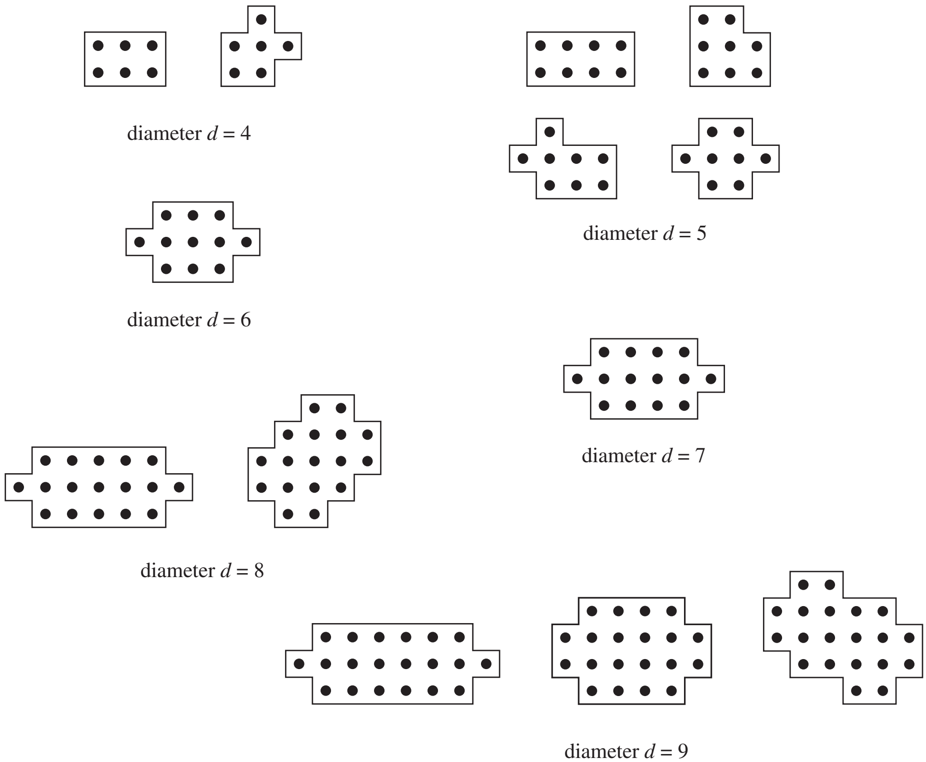

Fig. 12. Optimal quadristance anticodes in $\mathcal{G}_2^{\boxplus}$ of diameter $d = 4, 5, \dots, 9$.

Now let $\mathcal{A}_d$ denote an optimal (unrestricted) quadristance anticode of diameter d in $\mathcal{G}_2^{\boxplus}$. Arguing as in Lemma 10, it is easy to show that

$$\mathcal{A}_d = \bigcap_{z_1, z_2, z_3 \in \mathcal{A}_d} \mathcal{A}_d(z_1, z_2, z_3). \tag{88}$$

However, as can be seen from Fig. 11, the sets $\mathcal{A}_d(z_1, z_2, z_3)$ are no longer convex and their general shape (union of three octagons) is not preserved under intersection. Thus (88) and Theorem 36 do not suffice to determine the shape of $\mathcal{A}_d$. In fact, as we shall see in Fig. 12, $\mathcal{A}_d$ may come in several different shapes, at least for certain diameters.

Nevertheless, we can use an *arbitrary* shape in order to derive a *lower bound* on the cardinality of $\mathcal{A}_d$. Based on the available numerical evidence (cf. Fig. 12), we will use an octagon with equally truncated corners, namely the set of all $(x, y) \in \mathbb{Z}^2$ such that

$$0 \leqslant x \leqslant a, \quad c \leqslant x + y \leqslant a + b - c, \tag{89}$$

$$0 \leqslant y \leqslant b, \quad c - b \leqslant x - y \leqslant a - c. \tag{90}$$

We will denote such a set by $\mathcal{O}(a, b, c)$. We assume that each of the eight inequalities in (89) and (90) holds with equality for some point of $\mathcal{O}(a, b, c)$; otherwise, we can always re-parametrize accordingly. This, in particular, implies that

$$2c \leqslant \min\{a, b\}. \tag{91}$$

Note that the cardinality of $\mathcal{O}(a, b, c)$ is given by (14) with $c_0 = c_1 = c_2 = c_3 = c$. Thus the next step is to determine the quadristance diameter of $\mathcal{O}(a, b, c)$.

Lemma 37. *Let $\mathcal{O}(a, b, c)$ be the octagon in* (89) *and* (90) *and assume w.l.o.g. that $a \geqslant b$. Then the quadristance diameter of $\mathcal{O}(a, b, c)$ is given by*

$$d = a + 2b - 2c. \tag{92}$$

Proof. Let $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2)$, $z_3 = (x_3, y_3)$, $z_4 = (x_4, y_4)$ be four arbitrary points of $\mathcal{O}(a, b, c)$, and assume w.l.o.g. that $x_1 \leqslant x_2 \leqslant x_3 \leqslant x_4$. Let τ be a permutation such that $y_{\tau(1)} \leqslant y_{\tau(2)} \leqslant y_{\tau(3)} \leqslant y_{\tau(4)}$. If $\tau \in \Gamma$ so that $d_4(z_1, z_2, z_3, z_4)$ is given by (82), then

$$d_4(z_1, z_2, z_3, z_4) = \big(x_4 - x_1\big) + \big(y_{\tau(4)} - y_{\tau(1)}\big) \leqslant a + b \leqslant a + 2b - 2c,$$

where the first inequality follows from (89) and (90) while the second inequality follows from (91). Otherwise, $d_4(z_1, z_2, z_3, z_4)$ is given by (83) so that

$$\begin{aligned} d_4(z_1, z_2, z_3, z_4) &= \big(x_4 - x_1\big) + \big(y_{\tau(4)} - y_{\tau(1)}\big) + \min\big\{x_3 - x_2, y_{\tau(3)} - y_{\tau(2)}\big\} && (93)\\ &\leqslant \big(x_4 - x_1\big) + \Big(y_{\tau(4)} + y_{\tau(3)} - y_{\tau(2)} - y_{\tau(1)}\Big) && (94)\\ &= \big(x_4 \pm y_4\big) - \big(x_1 \pm y_1\big) + \big(\pm y_2 \pm y_3\big). && (95) \end{aligned}$$

There are four simple cases depending on the signs of y_1 and y_4 in (95). Observe that, in view of (94), exactly two of y_1, y_2, y_3, y_4 contribute to $d_4(z_1, z_2, z_3, z_4)$ with a positive sign and two with a negative sign. This immediately implies the following:

Case 1: $d_4(z_1, z_2, z_3, z_4) = (x_4 - y_4) - (x_1 + y_1) + (y_2 + y_3) \leqslant (a - c) - c + (b + b)$,

Case 2: $d_4(z_1, z_2, z_3, z_4) = (x_4 + y_4) - (x_1 - y_1) - (y_2 + y_3) \leqslant (a + b - c) - (c - b)$.

In the other two cases, y_1, y_4 contribute to $d_4(z_1, z_2, z_3, z_4)$ with opposite signs. Hence y_2, y_3 also have opposite signs, so that the last term in (95) is at most $|y_2 - y_3| \leqslant b$. Thus

Case 3: $d_4(z_1, z_2, z_3, z_4) \leqslant (x_4 - y_4) - (x_1 - y_1) + b \leqslant (a - c) - (c - b) + b$,

Case 4: $d_4(z_1, z_2, z_3, z_4) = (x_4 + y_4) - (x_1 + y_1) + b \leqslant (a + b - c) - c + b$.

The above shows that $d_4(z_1, z_2, z_3, z_4) \leqslant a + 2b - 2c$ for any $z_1, z_2, z_3, z_4 \in \mathcal{O}(a, b, c)$. To see that this bound holds with equality, consider the points $z_1 = (0, b - c)$, $z_2 = (c, 0)$, $z_3 = (a - c, b)$, $z_4 = (a, c)$. For these points $d_4(z_1, z_2, z_3, z_4) = a + 2b - 2c$ by (91) and Theorem 34. $\square$

Theorem 38. *Let $\mathcal{A}_d$ be an optimal quadristance anticode of diameter d in $\mathcal{G}_2^{\boxplus}$. Then*

$$|\mathcal{A}_d| \geqslant \left\lceil \frac{d^2 + 4d + 3}{6} \right\rceil = \left\lceil \frac{(d + 1)(d + 3)}{6} \right\rceil.$$

Proof. The lower bound follows by considering quadristance anticodes of type $\mathcal{O}(a, b, c)$. In view of (14), (91), and Lemma 37, the optimal parameters a, b, c are obtained by maximizing $|\mathcal{O}(a, b, c)| = (a + 1)(b + 1) - 2c(c + 1)$ subject to the constraints $a + 2b - 2c = d$ and $a \geqslant b \geqslant 2c$. The solution to this optimization problem is compiled in Table 5. $\square$

Table 5
Parameters of (conjecturally) optimal quadristance anticodes in $\mathcal{G}_2^{\boxplus}$

$d \pmod 6$	a	b	c	$\lvert\mathcal{O}(a,b,c)\rvert$
0	$\frac{2d+3}{3}$	$\frac{d+3}{3}$	$\frac{d}{6}$	$\frac{d^2+4d+6}{6}$
1	$\frac{2d+4}{3}$	$\frac{d+2}{3}$	$\frac{d-1}{6}$	$\frac{d^2+4d+7}{6}$
2	$\frac{2d+5}{3}$	$\frac{d+1}{3}$	$\frac{d-2}{6}$	$\frac{d^2+4d+6}{6}$
3	$\frac{2d+6}{3}$	$\frac{d}{3}$	$\frac{d-3}{6}$	$\frac{d^2+4d+3}{6}$
	$\frac{2d}{3}$	$\frac{d+3}{3}$	$\frac{d-3}{6}$	
4	$\frac{2d+1}{3}$	$\frac{d+2}{3}$	$\frac{d-4}{6}$	$\frac{d^2+4d+4}{6}$
5	$\frac{2d+2}{3}$	$\frac{d+1}{3}$	$\frac{d-5}{6}$	$\frac{d^2+4d+3}{6}$
	$\frac{2d+2}{3}$	$\frac{d+4}{3}$	$\frac{d+1}{6}$	

We have also used exhaustive computer search to find optimal quadristance anticodes in $\mathcal{G}_2^{\boxplus}$ of diameters up to $d = 9$. The results of this search are presented in Fig. 12, which shows *all* the optimal anticodes (up to obvious isomorphisms) for $d = 4, 5, \ldots, 9$. For $d = 3$, the optimal quadristance anticodes are simply the five tetromino shapes of [10].

Remark. We observe that—for the first time in this paper—optimal anticodes of a given diameter do not have a unique shape. Moreover, for $d = 3, 5, 9$ non-convex shapes occur among optimal quadristance anticodes, again for the first time in this paper.

Nevertheless, an octagon with equally truncated corners is always among the optimal shapes in Fig. 12. Hence the lower bound on $|\mathcal{A}_d|$ in Theorem 38 is exact at least up to $d = 9$. We conjecture that this bound is, in fact, exact for all diameters.

5. Applications of tristance and quadristance anticodes

Our study of tristance anticodes was originally motivated by applications to multi-dimensional interleaving [5,6,9,14]. A two-dimensional *interleaving scheme* $\mathcal{I}(t, r)$ of strength t with r repetitions is a labeling $\mathcal{I}(t, r) : \mathbb{Z}^2 \to \{1, 2, \ldots, \chi\}$ such that no integer in the range $\{1, 2, \ldots, \chi\}$ of $\mathcal{I}(t, r)$ appears more than r times among the labels of any connected subgraph of $\mathcal{G}_2^{\boxplus}$ with $\leqslant t$ vertices. The integer χ is called the *interleaving degree* of $\mathcal{I}(t, r)$ and denoted $\deg \mathcal{I}(t, r)$. Interleaving schemes of this kind may be used for error-control in optical, holographic, and magnetic recording [5,6]. The graphs $\mathcal{G}_2^{\infty}$, $\mathcal{G}_2^{\hexagon}$, and $\mathcal{G}_3^{\boxplus}$ are also relevant for these applications. In each case, the goal is to minimize the interleaving degree for a given strength t and a given number of repetitions r. Usually, the values of interest are when r is small (say $r = 1, 2, 3$) and t is large. We note that for $r = 1$, the problem has been completely solved in [6]. For $r = 2, 3$, upper bounds on the interleaving degree are given in [9]. In particular, it is shown in [9] that there exist interleaving schemes $\mathcal{I}(t, 2)$ with $\deg \mathcal{I}(t, 2) = (3/16)t^2 + O(t)$, and $\mathcal{I}(t, 3)$ with $\deg \mathcal{I}(t, 3) = (8/81)t^2 + O(t)$.

Our results on tristance anticodes provide *lower bounds* on the minimum possible interleaving degree of $\mathcal{I}(t, 2)$ as follows. If $\mathcal{A}_d$ is a tristance anticode of diameter $d = t - 1$, then any three points in $\mathcal{A}_d$ belong to a connected subgraph with $\leqslant t$ vertices, by definition. Therefore, no integer in the range of $\mathcal{I}(t,2)$ can be used more than twice in labeling the points of $\mathcal{A}_{t-1}$. Hence $\deg\mathcal{I}(t,2) \geqslant |\mathcal{A}_{t-1}|/2$. In conjunction with Theorems 15, 20, 27, this immediately implies that

$$\deg\mathcal{I}(t,2) \geqslant \begin{cases} \left\lceil \dfrac{t(t+1)}{7} \right\rceil & \text{in the grid graph } \mathcal{G}_2^{\boxplus}, \\[2ex] \left\lceil \dfrac{t(t+1)}{6} \right\rceil & \text{in the hexagonal graph } \mathcal{G}_2^{\hexagon}, \\[2ex] \left\lceil \dfrac{2t^2-1}{7} \right\rceil & \text{in the infinity graph } \mathcal{G}_2^{\infty}. \end{cases} \tag{96}$$

For the grid graph $\mathcal{G}_2^{\boxplus}$, a better bound was recently given in [14] in the case where t is even. In fact, it is shown in [14] that for even t the minimum possible interleaving degree of $\mathcal{I}(t,2)$ in $\mathcal{G}_2^{\boxplus}$ is exactly $\lfloor (3t^2+4)/16 \rfloor$ (the problem is still open for odd t). For $\mathcal{G}_2^{\hexagon}$ and $\mathcal{G}_2^{\infty}$, the bounds in (96) are the best known. Using similar reasoning, Theorem 38 implies that

$$\deg\mathcal{I}(t,3) \geqslant \left\lceil \frac{t(t+2)}{18} \right\rceil$$

in $\mathcal{G}_2^{\boxplus}$. This bound is also the best known (cf. [5]). Finally, the results of §4.1 herein imply a lower bound on $\deg\mathcal{I}(t,2)$ in the three-dimensional grid graph $\mathcal{G}_3^{\boxplus}$. For this graph, no upper bounds are yet known and even the problem of determining $\deg\mathcal{I}(t,1)$ is still open.

Another interesting application of tristance and quadristance anticodes is related to multicasting in processor networks. The general topology of such a network is often described by one of the graphs $\mathcal{G}_2^{\boxplus}$, $\mathcal{G}_2^{\hexagon}$, $\mathcal{G}_2^{\infty}$, $\mathcal{G}_3^{\boxplus}$, $\mathcal{G}_3^{\infty}$ (specifically, $\mathcal{G}_2^{\boxplus}$ and $\mathcal{G}_3^{\boxplus}$ are known as *mesh networks*, while $\mathcal{G}_2^{\hexagon}$ is sometimes called the *hexagonal grid network*). Each processor is viewed as a vertex in the graph capable of exchanging messages only with its neighbors. When a given processor needs to broadcast a message to k other processors (this is called *multicasting*), the most efficient solution is to send the message over the edges of a minimal spanning tree for the $k+1$ processors. This minimizes the total number of hops, where a *hop* is defined as communication between two neighboring processors. In general, then, how would one place the largest number of processors in such a network so that any one can multicast to any other two (respectively, any other three) with at most d hops? The answer is precisely the tristance anticode $\mathcal{A}_d$ (respectively, the quadristance anticode $\mathcal{A}_d$) in the corresponding graph. Given a specific source processor P_0, asking what is the largest set of processors such that P_0 can multicast to any two of them, or any three of them, with at most d hops gives rise to centered tristance, or quadristance, anticodes.

Our results for the grid graph $\mathcal{G}_2^{\boxplus}$ also have applications to the game of Go. Indeed, the game is played on a 19×19 square subgraph of $\mathcal{G}_2^{\boxplus}$, called the goban. Two players—Black and White—alternate moves, each move consisting of one stone of the player's color being placed on one of the 361 vertices of the goban. A set of stones of the same color is considered a connected group if the induced subgraph of $\mathcal{G}_2^{\boxplus}$ is connected. Thus our results in §2.2, §2.3, and §4.2 answer the following questions:

- How should three stones be played on an empty goban, so that they can then be all connected with at most k moves?
- Given a stone, how should two stones be played on an empty goban, so that all three stones can then be connected with at most k moves?
- Given two stones, where could one play a third stone so that all three can then be connected with at most k moves?
- Given three stones, where could one play a fourth stone so that all four can then be connected with at most k moves?

The answers to these questions are, respectively, the tristance anticode $\mathcal{A}_{k+2}$ given in Theorem 15, the centered tristance anticode $\mathcal{A}_{k+2}(z_0)$ in Theorem 7, the centered tristance anticode $\mathcal{A}_{k+2}(z_1, z_2)$ in Theorem 8, and the centered quadristance anticode $\mathcal{A}_{k+3}(z_1, z_2, z_3)$ in Theorem 36. It is interesting that the answers to the third and fourth questions above are drastically different (compare Figs. 2 and 11), even though the questions themselves appear to be similar. Of course, all these results assume an empty goban and no active opposition to the desired connection. Nevertheless, they could be of interest for computer Go applications [15]. We also have an algorithmic solution (to be presented elsewhere) for the case where the goban already has black and white stones in arbitrary positions.

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References

[1] R. Ahlswede, L.H. Khachatrian, The complete nontrivial-intersection theorem for systems of finite sets, J. Combin. Theory Ser. A 76 (1996) 121–138.
[2] R. Ahlswede, L.H. Khachatrian, The diametric theorem in Hamming spaces—optimal anticodes, Adv. Appl. Math. 20 (1998) 429–449.
[3] E. Bannai, Codes in bi-partite distance-regular graphs, J. London Math. Soc. 16 (1977) 197–202.
[4] N.L. Biggs, Perfect codes in graphs, J. Combin. Theory Ser. B 15 (1973) 289–296.
[5] M. Blaum, J. Bruck, P.G. Farrell, Two-dimensional interleaving schemes with repetitions, Technical Report RJ 10047 (90543), IBM Corporation, 1996.
[6] M. Blaum, J. Bruck, A. Vardy, Interleaving schemes for multidimensional cluster errors, IEEE Trans. Inform. Theory 44 (1998) 730–743.
[7] J.H. Conway, N.J.A. Sloane, Sphere Packings, Lattices, and Groups, Springer, New York, 1988.
[8] Ph. Delsarte, An algebraic approach to the association schemes of coding theory, Philips J. Res. 10 (1973) 1–97.
[9] T. Etzion, A. Vardy, Two-dimensional interleaving schemes with repetitions: constructions and bounds, IEEE Trans. Inform. Theory 48 (2002) 428–457.
[10] S.W. Golomb, Polyominoes, second ed., Princeton University Press, Princeton, 1994.
[11] J. Kratochvíl, Perfect codes over graphs, J. Combin. Theory 40 (1986) 224–228.
[12] F.J. MacWilliams, N.J.A. Sloane, The Theory of Error Correcting Codes, Elsevier, Amsterdam, 1977.

[13] W.J. Martin, X.J. Zhu, Anticodes for the Grassman and bilinear forms graphs, Designs Codes Crypt. 6 (1995) 73–79.

[14] Y. Merksamer, T. Etzion, On the optimality of coloring with a lattice, SIAM J. Discrete Math., (2005), to appear.

[15] M. Müller, Computer Go: A research agenda, Internat. Comput. Games Assoc. J. 22 (1999) 104–112.

[16] V.S. Pless, W.C. Huffman (Eds.), Handbook of Coding Theory, Elsevier, Amsterdam, 1998.

[17] M. Schwartz, T. Etzion, Codes and anticodes in the Grassman graph, J. Combin. Theory Ser. A 97 (2002) 27–42.

[18] M. Schwartz, T. Etzion, Optimal two-dimensional 3-dispersion lattices, Lecture Notes in Computer Science, vol. 2643, 2003, pp. 216–225.