On Optimal Locally Repairable Codes and Generalized Sector-Disk Codes

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***Abstract*—Optimal locally repairable codes with information locality are considered. Optimal codes are constructed, whose length is also order-optimal with respect to a new bound on the code length derived in this article. The length of the constructed codes is super-linear in the alphabet size, which improves upon the well known pyramid codes, whose length is only linear in the alphabet size. The recoverable erasure patterns are also analyzed for the new codes. Based on the recoverable erasure patterns, we construct generalized sector-disk (GSD) codes, which can recover from disk erasures mixed with sector erasures in a more general setting than known sector-disk (SD) codes. Additionally, the number of sectors in the constructed GSD codes is super-linear in the alphabet size, compared with known SD codes, whose number of sectors is only linear in the alphabet size.**

***Index Terms*—Distributed storage, locally repairable codes, sector-disk codes, Goppa codes.**

I. Introduction

In the large distributed storage systems of today, disk failures are the norm rather than the exception. Thus, erasure-coding techniques are employed to protect the data from disk failures. An $[n,k]$ storage code encodes k information symbols to n symbols and stores them across n disks in a storage system. Generally speaking, among all storage codes, maximum distance separable (MDS) codes are preferred for practical systems both in terms of redundancy and in terms of reliability. However, as pointed in [40], MDS codes such as Reed-Solomon codes suffer from a high repair cost. This is mainly because, for an $[n,k]$ MDS code, whenever one wants to recover a symbol, one needs to contact k surviving symbols, which is costly, especially in large-scale distributed file systems.

To improve the repair efficiency, locally repairable codes, such as pyramid codes [24], are deployed to reduce the number of symbols contacted during the repair process. More precisely, the concept of r-locality for a code $\mathcal{C}$ was initially studied in [19] to ensure that a failed symbol can be recovered by only accessing $r \ll k$ other symbols which form a repair set.

In the past decade, the original definition has been generalized in different aspects. Firstly, to guarantee that the system can recover locally from multiple erasures, the notion of r-locality was generalized to (r,δ)-locality, namely, each repair set is capable of recovering from $\delta-1$ erasures. Secondly, to let code symbols have good availability, the notion of locality has been generalized to (r,δ)-availability [37] (or $(r,\delta)_c$-locality [46]), in which case a code symbol has more than one repair set. Thus, each repair set can be viewed as a backup for the target code symbol, hence the code symbol can be accessed independently through each repair set. Finally, to satisfy differing locality requirements, the notion of locality has been generalized to the hierarchical and the unequal locality cases. Upper bounds on the minimum Hamming distance of locally repairable codes and constructions for them have been reported in the literature for those generalizations. For examples, the reader may refer to [4], [6], [8], [11], [13], [24], [25], [30], [31], [33], [35], [36], [42], [43], [45], [47] for (r,δ)-locality, [9], [10], [23], [37], [41], [44], [46] for (r,δ)-availability, [39] for hierarchical locality, and [28], [48] for unequal locality.

Based on the observation given in [18], locally repairable codes may recover from some special erasure patterns beyond their minimum Hamming distance. Thus, another research branch for locally repairable codes is the study of their recoverable erasure patterns. In this aspect, two special kinds of codes have received most of the attention. One is the $(\delta-1,\gamma)$-maximally recoverable code first introduced in [5], [18], that can recover from erasure patterns that include any $\delta-1$ erasures from each repair set, and any other γ erasures. The $(\delta-1,\gamma)$-maximally recoverable codes are equivalent to $(\delta-1,\gamma)$-partial MDS codes a special kind of array codes that was introduced to improve the storage efficiency of redundant arrays of independent disks (RAIDs) [5]. The other is $(\delta-1,\gamma)$-sector-disk (SD) codes [34] that can recover from erasure patterns that include any $\delta-1$ erasures from each repair set with consistent indices (i.e., whole disk erasures) and any other γ erasures (i.e., sector erasures). For construction of SD codes the reader may refer to [5], [7], [12], [17], [18], [33], [34] for example. The main drawback of all of the reported constructions for SD codes is the requirement for a large finite field.

In this article, we focus on both (r,δ)-locality and recoverable erasure patterns beyond the minimum Hamming distance. For (r,δ)-locality we propose constructions of locally repairable codes whose information symbols have (r,δ)-locality and their length is super-linear in the field size. The codes generated by our constructions have new parameters compared with known locally repairable codes. In particular, our codes have a smaller requirement on the field size compared with the well-known pyramid codes [24] and Tamo-Barg codes [43] for example. Additionally, we consider

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the following fundamental problem: how long can a locally repairable codes be, whose information symbols have (r, δ)-locality? We propose a new upper bound on the length of optimal locally repairable codes. Based on this bound, we prove that the codes generated by our construction may have order-optimal length. We also analyze recoverable erasure patterns beyond the minimum Hamming distance in the codes we construct. Based on this analysis, we construct array codes that can recover special erasure patterns which mix whole disk erasures together with additional sector erasures that beyond the minimum Hamming distance. These codes generalize SD codes, and we therefore call them generalized sector-disk (GSD) codes. Finally, the classic Goppa codes are modified into locally repairable codes. In this way, the generated codes not only share similar parameters with the ones in [11], but also yield optimal locally repairable codes with new parameters.

The remainder of this article is organized as follows. Section II introduces some necessary notation and results. Section III proposes a new construction of locally repairable codes and a bound on code length. In Section IV we introduce GSD codes, and in Section V we modify classical Goppa codes into a class of locally repairable codes. Section VI concludes this article with some remarks.

II. Preliminaries

Throughout this article, the following notation are used:

- For a positive integer n, let $[n]$ denote the set $\{1, 2, \cdots, n\}$;
- For any prime power q, let $\mathbb{F}_q$ denote the finite field with q elements;
- An $[n, k]_q$ linear code $\mathcal{C}$ over $\mathbb{F}_q$ is a k-dimensional subspace of $\mathbb{F}_q^n$ with a $k \times n$ generator matrix $G = (\mathbf{g}_1, \mathbf{g}_2, \cdots, \mathbf{g}_n)$, where $\mathbf{g}_i$ is a column vector of length k for all $1 \leqslant i \leqslant n$. Specifically, it is called an $[n, k, d]_q$ linear code if the minimum Hamming distance is d;
- For a subset $S \subseteq [n]$, let $|S|$ denote the cardinality of S, $\mathrm{Span}(S)$ be the linear space spanned by $\{\mathbf{g}_i : i \in S\}$ over $\mathbb{F}_q$ and $\mathrm{Rank}(S)$ be the dimension of $\mathrm{Span}(S)$.

A. Locally Repairable Codes

Let us recall some necessary definitions concerning locally repairable codes. Throughout this article we assume that $\mathcal{C}$ be an $[n, k, d]_q$ linear code with generator matrix $G = (\mathbf{g}_1, \mathbf{g}_2, \cdots, \mathbf{g}_n)$.

Definition 1 ([24], [35]): The ith code symbol of an $[n, k, d]_q$ linear code $\mathcal{C}$, is said to have (r, δ)-locality if there exists a subset $S_i \subseteq [n]$ (a *repair set*) such that

- $i \in S_i$ and $|S_i| \leqslant r + \delta - 1$; and
- The minimum Hamming distance of the punctured code $\mathcal{C}|_{S_i}$, obtained by deleting the code symbols c_j for all $j \in [n] \setminus S_i$, is at least δ.

Furthermore, an $[n, k, d]_q$ linear code $\mathcal{C}$ is said to have information (r, δ)-locality (denoted as $(r, \delta)_i$-locality) if there exists a k-subset $I \subseteq [n]$ with $\mathrm{Rank}(I) = k$ such that for each $i \in I$, the ith code symbol has (r, δ)-locality and all symbol (r, δ)-locality (denoted as $(r, \delta)_a$-locality) if all the n code symbols have (r, δ)-locality.

In [35] (also, [19] for $\delta = 2$), an upper bound on the minimum Hamming distance of linear codes with $(r, \delta)_i$-locality was derived as follows.

Lemma 1 ([35]): The minimum distance of an $[n, k, d]_q$ code $\mathcal{C}$ with $(r, \delta)_i$-locality is upper bounded by

$$d \leqslant n - k + 1 - \left(\left\lceil \frac{k}{r} \right\rceil - 1 \right) (\delta - 1). \tag{1}$$

Definition 2: A linear code with $(r, \delta)_i$-locality is said to be an *optimal locally repairable code* if its minimum Hamming distance meets the Singleton-type bound of Lemma 1 with equality.

According to (1), even for an optimal $[n, k, d]_q$ linear code with $(r, \delta)_i$-locality (or $(r, \delta)_a$-locality), $d < n - k + 1$ under the nontrivial case $k > r$. Thus, for a linear code with $(r, \delta)_i$-locality, it is natural to ask if it is possible for an erasure pattern $E \subset [n]$ with size $d \leqslant |E| \leqslant n - k$ to be recoverable [19]. Generally this problem is still open. However, two special settings of this problem received special attention in previous works.

Setting I: (e.g., [5], [34]) For a linear code with $(r, \delta)_a$-locality, let $(r + \delta - 1)|n$ and $|\{S_i : i \in [n]\}| = \frac{n}{r+\delta-1}$, i.e., all the n symbols are divided into $\frac{n}{r+\delta-1}$ repair sets. Let $s = \frac{n}{r+\delta-1} r - k$ and assume the elements of S_i are denoted by $\{s_{i,1}, s_{i,2}, \ldots, s_{i,r+\delta-1}\}$. An erasure pattern E is required to be recoverable if there exists a $(\delta - 1)$-subset of $[r + \delta - 1]$, $\{j_1, j_2, \cdots, j_{\delta-1}\}$, and there exists a set $E^* \subseteq E \subseteq [n]$, $|E^*| \leqslant s$, and

$$(E \backslash E^*) \cap S_i \subseteq \{s_{i,j_1}, s_{i,j_2}, \ldots, s_{i,j_{\delta-1}}\} \text{ for each } i \in [n].$$

Setting II: (e.g., [18]) For a linear code with $(r, \delta)_a$-locality, let $(r + \delta - 1)|n$ and $|\{S_i : i \in [n]\}| = \frac{n}{r+\delta-1}$, i.e., all the n symbols are divided into $\frac{n}{r+\delta-1}$ repair sets. Let $s = \frac{n}{r+\delta-1} r - k$. An erasure pattern E is required to be recoverable if there exists a set $E^* \subseteq E \subseteq [n]$, $|E^*| \leqslant s$ and

$$|(E \backslash E^*) \cap S_i| \leqslant \delta - 1 \text{ for each } 1 \leqslant i \leqslant \frac{n}{r + \delta - 1}.$$

Definition 3: An $[n, k, d]_q$ linear code that satisfies the conditions of Setting I is said to be a sector-disk code ($(\delta - 1, s)$-SD).

As an intuition, we make the following analogies between a distributed storage system and Setting I. In this analogy, we have a total of $r + \delta - 1$ disks, each containing $\frac{n}{r+\delta-1}$ sectors, with a total number of sectors in the system which is n. The ith stripe, i.e., the set containing the ith sector from each disk, is an (r, δ)-repair set, for each i. Finally, an SD code is capable of correcting $\delta - 1$ whole disk erasures, as well as an extra s erased sectors.

Definition 4: An $[n, k, d]_q$ linear code that satisfies the conditions of Setting II is said to be a maximally recoverable code ($(\delta - 1, s)$-MR code).

In this article we study codes from Setting I, whereas Setting II is mentioned for completeness and for comparison. In Fig. 1, we list five different types of erasure patterns for $(1, 2)$-MR codes, where MR, SD, and LRC denote $(1, 2)$-MR codes, $(1, 2)$-SD codes, and optimal locally repairable codes (LRC) with parameters $[12, 8, 3]_q$ and $(3, 2)_a$-locality, respectively.

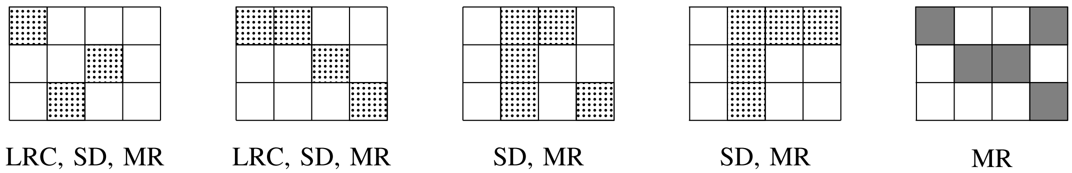

Fig. 1. Different types of erasure patterns recoverable for LRC, SD, and MR codes.

MR codes are also known as partial MDS (PMDS) codes [5], [7], [12]. It is easy to check that Setting I is a special case of Setting II, thus, MR codes are also SD codes, but not vice versa. For explicit constructions, the reader may refer to [7], [34] for SD codes, and [5], [7], [12], [17], [33] for MR codes. Finally, another example of a family of codes that may recover special erasure patterns beyond the minimum Hamming distance is STAIR codes [29].

B. Packings and Steiner Systems

We now turn to describe some definitions and known facts concerning the combinatorial objects of packings and Steiner systems.

Definition 5 ([14], VI. 40): Let $n \geqslant 2$ and t, τ be positive integers. A τ-$(n,t,1)$-*packing* is a pair $(X, \mathcal{B})$, where X is a set of n elements (called points) and $\mathcal{B} \subseteq 2^X$ is a collection of t-subsets of X (called blocks), such that each τ-subset of X is contained in at most one block of $\mathcal{B}$. If $\tau = 2$, it is also denoted as an $(n,t,1)$-packing. The packing is said to be *regular* if each element of X appears in exactly w blocks, denoted as a w-regular τ-$(n,t,1)$-packing.

Definition 6 ([14], II. 5): Let $n \geqslant 2$ and t, τ be positive integers. A (τ, t, n)-*Steiner system* is a pair $(X, \mathcal{B})$, where X is a set of n elements (called points) and $\mathcal{B} \subseteq 2^X$ is a collection of t-subsets of X (called blocks), such that each τ-subset of X is contained in exactly one block of $\mathcal{B}$.

Lemma 2 ([14], II. 5): A (τ, t, n)-Steiner system is a $\frac{\binom{n-1}{\tau-1}}{\binom{t-1}{\tau-1}}$-regular τ-$(n,t,1)$-packing.

Remark 1: Given positive integers τ, t and n, the natural necessary conditions for the existence of a (τ, t, n)-Steiner system are that $\binom{t-i}{\tau-i} | \binom{n-i}{\tau-i}$ for all $0 \leqslant i \leqslant \tau - 1$. It was shown in [27] that these conditions are also sufficient except perhaps for finitely many cases.

III. Constructions of Locally Repairable Codes

In this section, we introduce a general construction of locally repairable codes with information locality. Let $k = r\ell + v$ with $0 < v \leqslant r$, $\ell = \lceil k/r \rceil - 1$, and $n = k + (\ell+1)(\delta - 1) + h$ with $h \geqslant 0$, where all parameters are integers. Herein, the parameter k will be used to denote the number of information symbols, r corresponds to locality, $\ell + 1$ denotes the number of pairwise disjoint repair sets, and δ means that the punctured code over each repair set has Hamming distance at lest δ.

Construction A: Let the k information symbols be partitioned into $\ell + 1$ sets, say,

$$I^{(i)} = (I_{i,1}, I_{i,2}, \ldots, I_{i,r}), \quad \text{for } i \in [\ell],$$
$$I^{(\ell+1)} = (I_{\ell+1,1}, I_{\ell+1,2}, \ldots, I_{\ell+1,v}).$$

Let S be an h-subset of $\mathbb{F}_q$ and denote $A \triangleq \mathbb{F}_q \backslash S$. Let $\mathcal{A} = \{A_i : 1 \leqslant i \leqslant \ell+1\}$ be a family of subsets of A with $|A_i| = r + \delta - 1$ for $1 \leqslant i \leqslant \ell$ and $|A_{\ell+1}| = v + \delta - 1$. Define

$$g_i(x) = \prod_{\theta \in A_i} (x - \theta) \text{ for } 1 \leqslant i \leqslant \ell + 1$$

and

$$\Delta(x) = \prod_{1 \leqslant i \leqslant \ell+1} g_i(x).$$

A linear code with length n can be generated by defining a linear map from the information $\boldsymbol{I} = (I_{1,1}, I_{1,2}, \ldots I_{1,r+\delta-1}, I_{2,1} \ldots, I_{\ell+1,1}, \ldots, I_{\ell+1,v}) \in \mathbb{F}_q^k$ to a codeword $\boldsymbol{C}(\boldsymbol{I}) = (c_{1,1}, \ldots, c_{\ell,r+\delta-1}, c_{\ell+1,1}, \ldots, c_{\ell+1,v+\delta-1}, c_{\ell+2,1}, \ldots, c_{\ell+2,h}) \in \mathbb{F}_q^n$, thus the $[n,k]_q$ linear code is $\mathcal{C} = \{\boldsymbol{C}(\boldsymbol{I}) : \boldsymbol{I} \in \mathbb{F}_q^k\}$. This mapping is performed by the following two steps:

a) Step 1: For $1 \leqslant j \leqslant \ell+1$, by polynomial interpolation, there exists a unique $f_j(x) \in \mathbb{F}_q[x]$ with $\deg(f_j) < |A_j| - \delta + 1 \leqslant r$ for $1 \leqslant j \leqslant \ell$ and $\deg(f_j) < |A_j| - \delta + 1 \leqslant v$ for $j = \ell + 1$ such that $f_j(\theta_{j,t}) = I_{j,t}$ for $1 \leqslant t \leqslant |A_j| - \delta + 1$, where $A_j = \{\theta_{j,t} : 1 \leqslant t \leqslant |A_j|\}$. For $1 \leqslant j \leqslant \ell + 1$ and $1 \leqslant t \leqslant |A_j|$, set $c_{j,t} = f_j(\theta_{j,t})$.

b) Step 2: Let

$$f_I(x) = \Delta(x) \sum_{1 \leqslant i \leqslant \ell+1} \frac{f_i(x)}{g_i(x)}. \tag{2}$$

Set $c_{\ell+2,i} = f_I(s_i)$ for $1 \leqslant i \leqslant h$, where $S = \{s_i : 1 \leqslant i \leqslant h\}$. Note that $\sum_{i=1}^{\ell+1} |A_i| = n - h$, which means we finally obtain a code with length n.

Example 1: Let $n = 10$, $k = 6$, $r = 2$, $\delta = 2$, $q = 8$, $\mathbb{F}_8 = \{\alpha_i : 0 \leqslant i \leqslant 7\}$, $S = \{\alpha_7\}$, and

$$\begin{aligned} \mathcal{A} = \{A_1 &= \{\theta_{1,1} = \alpha_0, \theta_{1,2} = \alpha_3, \theta_{1,3} = \alpha_2\}, \\ A_2 &= \{\theta_{2,1} = \alpha_1, \theta_{2,2} = \alpha_4, \theta_{2,3} = \alpha_3\}, \\ A_3 &= \{\theta_{3,1} = \alpha_3, \theta_{3,2} = \alpha_6, \theta_{3,3} = \alpha_5\}\}. \end{aligned}$$

By Construction A, we can generate a $[10, 6, 3]_8$ linear code with information $(2, 2)$-locality. In Fig. 2, we depict the underlying idea of our construction for this case, where, we use a line to denote a repair set, use dots to denote code symbols, use a curve to denote the polynomial f_I determining global parity checks and $a_{i,j}$, $1 \leqslant i, j \leqslant 3$ denotes a predetermined constant given by $a_{i,j} \triangleq \Delta(x)/g_i(x)|_{x=\theta_{i,j}}$ with $A_i = \{\theta_{i,1}, \theta_{i,2}, \theta_{i,3}\}$ for $1 \leqslant i \leqslant 3$ and $1 \leqslant j \leqslant 3$.

Lemma 3: The code $\mathcal{C}$ generated by Construction A is an $[n,k]_q$ linear code with $(r, \delta)_i$-locality.

Proof: In Construction A, since $\Delta(x)/g_i(x)$ for $1 \leqslant i \leqslant \ell+1$ is independent with the information I, it is easy to verify that $\mathcal{C}$ is an $[n,k]_q$ linear code. By Construction A, Step 1, for any $C \in \mathcal{C}$ and $1 \leqslant i \leqslant \ell + 1$, $(c_{i,1}, c_{i,1}, \ldots, c_{i,|A_i|})$ is the evaluation of a polynomial with degree at most $|A_i| - \delta$,

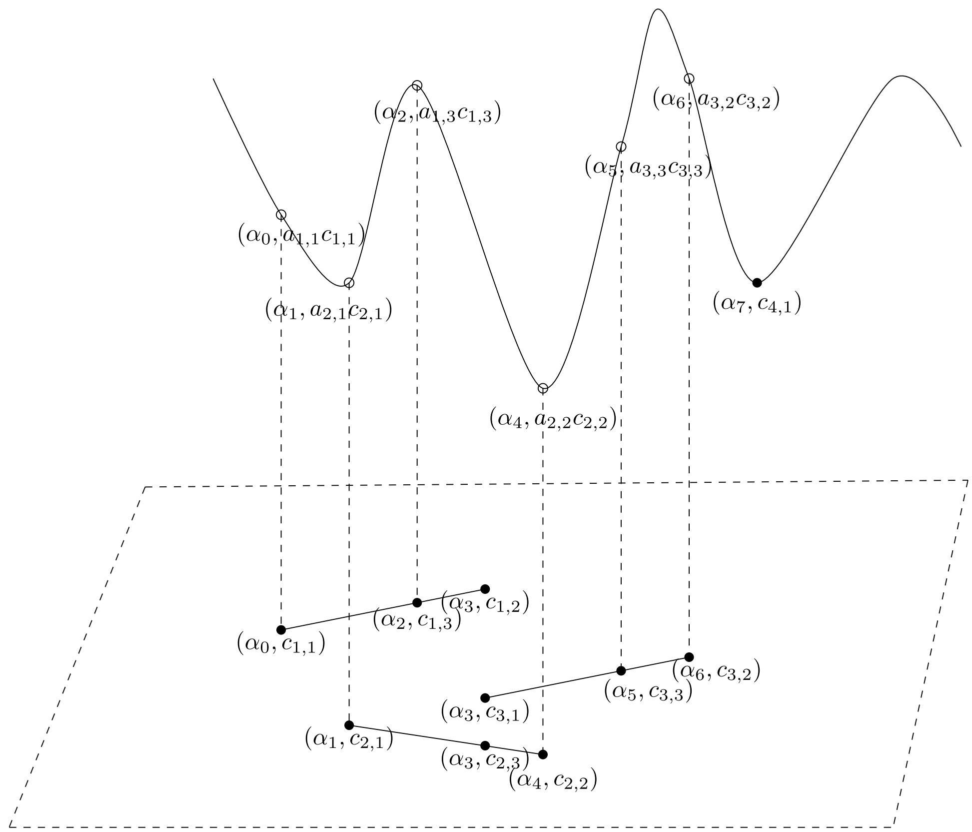

Fig. 2. A $[10, 6, 3]_8$ linear code with information $(2, 2)$-locality constructed by Construction A.

which means any $|A_i| - \delta + 1 \leqslant r$ components are capable of recovering the remaining components. Thus, the code $\mathcal{C}$ has $(r, \delta)_i$-locality. ■

For ease of presentation, we use the evaluation points (instead of the indices of code symbols) to denote erasure patterns. Additionally, we shall group the erased positions by the index of the repair set they hit. Namely, we shall use $\mathcal{E} = \{E_1, \dots, E_{\ell+2}\}$ to denote an erasure pattern, where $E_j \subseteq A_j$ is the set of erasure points in A_j, $1 \leqslant j \leqslant \ell + 1$, and $E_{\ell+2} \subseteq S$ is the set of erasure points in S.

Theorem 1: Let $\mathcal{C}$ be the linear code generated by Construction A. Assume $\mathcal{E} = \{E_i \ : \ 1 \leqslant i \leqslant \ell+2\}$ is an erasure pattern, with $E_i \subseteq A_i$ for $1 \leqslant t \leqslant \ell + 1$ and $E_{\ell+2} \subseteq S$. Suppose $|E_i| \geqslant \delta$ for $i \in \{i_t\colon 1 \leqslant t \leqslant w\} \subseteq [\ell + 1]$ and $|E_i| \leqslant \delta - 1$ otherwise. If the erasure pattern $\mathcal{E}$ satisfies

$$\left|\bigcup_{1\leqslant t\leqslant w} E_{i_t}\right| + |E_{\ell+2}| \leqslant h + \delta - 1, \tag{3}$$

and for any $1 \leqslant j \leqslant w$

$$\left|A_{i_j} \cap \left(\bigcup_{j\neq t\in[w]} A_{i_t}\right)\right| \leqslant \delta - 1, \tag{4}$$

then the erasure pattern $\mathcal{E}$ can be recovered.

Remark 2: Before proving Theorem 1, we want to highlight that the size $\left|(\bigcup_{1\leqslant t\leqslant w} E_{i_t}) \cup E_{\ell+2}\right|$ dictates whether an erasure pattern is recoverable, and not the number of erased coordinates, i.e., $\sum_{1\leqslant t\leqslant w} |E_{i_t}| + |E_{\ell+2}|$. This is to say, if there are erasures that share the same evaluation point (even in different coordinates), then those erasures as a whole will only increase the discriminant value by one. In such a case we may recover more than $h + \delta - 1$ erasures that are guaranteed to be recoverable by the value of the Singleton-type bound, i.e., $h + \delta$.

Proof: Since the linear code generated by Construction A has $(r, \delta)_i$-locality, the locality is capable of recovering all the erasures for the case $E_i \in \mathcal{E}$ with $|E_i| \leqslant \delta - 1$ and $1 \leqslant i \leqslant \ell + 1$ independently. Thus, in this proof we only need to consider the case $E_i \in \mathcal{E}$ with $|E_i| \geqslant \delta$ and $1 \leqslant i \leqslant \ell + 1$, i.e., E_{i_t} for $1 \leqslant t \leqslant w$. To recover the erasures, by Construction A, it is sufficient to recover $f_I(x)$ and $f_{i_t}(x)$ for $1 \leqslant t \leqslant w$. By (2), $f_I(x)$ is determined by $f_j(x)$ for $1 \leqslant j \leqslant \ell + 1$. However, we only know a part of them, i.e., $f_j(x)$ for $j \in [\ell+1] \setminus \{i_t : 1 \leqslant t \leqslant w\}$. Thus, we rewrite $f_I(x)$ and extract the unknown part denoted as $f_E(x)$.

Let

$$\begin{aligned}\Phi(x) &\triangleq \gcd\left(\frac{\Delta(x)}{g_{i_1}(x)}, \frac{\Delta(x)}{g_{i_2}(x)}, \cdots, \frac{\Delta(x)}{g_{i_w}(x)}\right) \\ &= \frac{\Delta(x)}{\operatorname{lcm}(g_{i_1}(x), g_{i_2}(x), \dots, g_{i_w}(x))} \\ &= \frac{\Delta(x)}{\prod_{\theta\in U}(x-\theta)},\end{aligned}$$

where $U \triangleq \bigcup_{1\leqslant t\leqslant w} A_{i_t}$. Considering $f_I(x)$ in Construction A, it can be rewritten as

$$f_I(x) = \Delta(x) \sum_{1\leqslant i\leqslant \ell+1} \frac{f_i(x)}{g_i(x)}$$

$$\begin{aligned}
&=\Delta(x)\sum_{i\in\{i_t\,:\,1\leqslant t\leqslant w\}}\frac{f_i(x)}{g_i(x)}+\Delta(x)\sum_{i\in[\ell+1]\setminus\{i_t\,:\,1\leqslant t\leqslant w\}}\frac{f_i(x)}{g_i(x)}\\
&=\Phi(x)\sum_{1\leqslant t\leqslant w}f^*_{i_t}(x)+g(x)\\
&=\Phi(x)f_E(x)+g(x), \qquad (5)
\end{aligned}$$

where

$$g(x)=\Delta(x)\sum_{j\in[\ell+1]\setminus\{i_t\,:\,1\leqslant t\leqslant w\}}\frac{f_j(x)}{g_j(x)}$$

is a known polynomial determined by the known code symbols by Construction A,

$$f^*_{i_t}(x)=\frac{\Delta(x)f_{i_t}(x)}{\Phi(x)g_{i_t}(x)}=\frac{f_{i_t}(x)\prod_{\theta\in U}(x-\theta)}{g_{i_t}(x)}, \qquad (6)$$

and

$$\deg(f^*_{i_t}(x))=|U|-|A_{i_t}|+\deg(f_{i_t}(x)) \qquad (7)$$

for $1\leqslant t\leqslant w$. Now the remaining part of the proof is mainly two steps, first, we need to show $f_E(x)$ can be recoverable, which means $f_I(x)$ is recoverable, and then $f_{i_t}(x)$ for $1\leqslant t\leqslant w$ is recoverable, i.e., I is recoverable.

Recall that (4) means that for $1\leqslant t\leqslant w$ there exists a set $A^*_{i_t}\triangleq A_{i_t}\setminus\left(\bigcup_{\tau\neq t,1\leqslant\tau\leqslant w}A_{i_\tau}\right)$ such that $|A^*_{i_t}|\geqslant r$. Let

$$e_{i_t,j}\triangleq\left.\frac{\prod_{\theta\in U}(x-\theta)}{g_{i_t}(x)}\right|_{x=\theta_{i_t,j}} \qquad (8)$$

for $1\leqslant t\leqslant w$ and $\theta_{i_t,j}\in A_{i_t}$. Then, for $\theta_{i_t,j}\in A^*_{i_t}$, we have

$$e_{i_t,j}c_{i_t,j}=e_{i_t,j}f_{i_t}(\theta_{i_t,j})=f^*_{i_t}(\theta_{i_t,j})=f_E(\theta_{i_t,j}), \qquad (9)$$

where the last equality holds by the fact that $f^*_{i_\tau}(\theta_{i_t,j})=0$ for $1\leqslant\tau\leqslant w$, $\tau\neq t$, and $\theta_{i_\tau,j}\in A^*_{i_t}$. Let $U^*=\bigcup_{1\leqslant t\leqslant w}A^*_{i_t}$. Note that for $\theta\in U\setminus U^*$ and $1\leqslant t\leqslant w$, $f^*_{i_t}(\theta)=0$ if $\theta\notin A_{i_t}$. For $\theta\in U\setminus U^*$, by (6) we have

$$\begin{aligned}
\sum_{\substack{\theta_{i_t,j}=\theta\in A_{i_t},\\1\leqslant t\leqslant w}}e_{i_t,j}c_{i_t,j}&=\sum_{\substack{\theta_{i_t,j}=\theta\in A_{i_t},\\1\leqslant t\leqslant w}}e_{i_t,j}f_{i_t}(\theta_{i_t,j})\\
&=\sum_{\substack{\theta_{i_t,j}=\theta\in A_i,\\1\leqslant t\leqslant w}}f^*_i(\theta_{i_t,j})\\
&=f_E(\theta),
\end{aligned}$$

where $e_{i_t,j}$ is defined by (8). The last equation implies that if we know all the code symbols in A_{i_t} for $1\leqslant t\leqslant w$ corresponding to the same element $\theta\in U\setminus U^*$ then we know the value of $f_E(\theta)$. In other words, we know all the values $f_E(\theta)$ for $\theta\in(U\setminus U^*)\setminus\left(\bigcup_{1\leqslant t\leqslant w}E_{i_t}\right)$.

Furthermore, for $\theta=\theta_{\ell+2,t}\in S\setminus E_{\ell+2}$, we have $c_{\ell+2,t}=f_I(\theta)=\Phi(\theta)f_E(\theta)+g(\theta)$, i.e., $f_E(\theta)=\frac{c_{\ell+2,t}-g(\theta)}{\Phi(\theta)}$, where $g(x)$ can be regarded as a known polynomial. Thus, under the erasure pattern $\mathcal{E}$, we know

$$\begin{aligned}
&\left|U^*\setminus\left(\bigcup_{1\leqslant t\leqslant w}E_{i_t}\right)\right|+\left|(U\setminus U^*)\setminus\left(\bigcup_{1\leqslant t\leqslant w}E_{i_t}\right)\right|\\
&+|S|-|E_{\ell+2}|\\
=&\left|U^*\setminus\left(\bigcup_{1\leqslant t\leqslant w}E_{i_t}\right)\right|+\left|(U\setminus U^*)\setminus\left(\bigcup_{1\leqslant t\leqslant w}E_{i_t}\right)\right|\\
&+h-|E_{\ell+2}|\\
\geqslant&\left|U^*\setminus\left(\bigcup_{1\leqslant t\leqslant w}E_{i_t}\right)\right|+\left|(U\setminus U^*)\setminus\left(\bigcup_{1\leqslant t\leqslant w}E_{i_t}\right)\right|\\
&+\left|\left(\bigcup_{1\leqslant t\leqslant w}E_{i_t}\right)\right|-\delta+1\\
=&|U|-\delta+1\\
>&|U|-|A_i|+\deg(f_{i_t}(x))=\deg(f^*_{i_t}(x))
\end{aligned}$$

evaluation points and the corresponding value for $f_E(x)$, where the two inequalities hold by (3) and (7), respectively. This is to say that we can recover $f_E(x)$ since $\deg(f_E(x))\leqslant\max_{1\leqslant t\leqslant w}\deg(f^*_{i_t}(x))$. By (5), the fact that $g(x)$ and $\Phi(x)$ are known polynomials means that we can recover $f_I(x)$ and thus also the code symbols in $E_{\ell+2}$. Recall that for $\theta_{i_t,j}\in A^*_{i_t}$ $1\leqslant t\leqslant w$, (9) means that $f_{i_t}(\theta_{i_t,j})=\frac{f_E(\theta_{i_t,j})}{e_{i_t,j}}$, where we apply the fact that $e_{i_t,j}\neq 0$ for $\theta_{i_t,j}\in A^*_{i_t}$ according to (8). Thus, we know the value of $f_{i_t}(\theta)$ for $\theta\in A^*_{i_t}$ and $1\leqslant t\leqslant w$. Now the fact that

$$\left|A_{i_t}\cap\left(\bigcup_{1\leqslant j\leqslant w,j\neq t}A_{i_j}\right)\right|\leqslant\delta-1,$$

means $|A^*_{i_t}|\geqslant|A_{i_t}|-\delta+1>\deg(f_{i_t}(x))$ by Step 1 of Construction A. This is to say that we can also recover $f_{i_t}(x)$ for $1\leqslant t\leqslant w$ and all the code symbols in A_{i_t} for $1\leqslant t\leqslant w$, which completes the proof. ■

Corollary 1: If the set system $\mathcal{A}$ of Construction A satisfies that for any μ-subset D of $[\ell+1]$

$$\left|A_i\cap\left(\bigcup_{j\neq i,j\in D}A_j\right)\right|\leqslant\delta-1\quad\text{for } i\in D, \qquad (10)$$

then the code $\mathcal{C}$ generated by Construction A is an $[n,k,d]_q$ linear code with $(r,\delta)_i$ locality and $d\geqslant\min\{(\mu+1)\delta,h+\delta\}$. Furthermore, if $h+\delta\leqslant(\mu+1)\delta$, then the code $\mathcal{C}$ is optimal with respect to the bound in Lemma 1.

Proof: By Lemma 3, we only need to prove $d\geqslant\min\{(\mu+1)\delta,h+\delta\}$. To bound the minimum Hamming distance of $\mathcal{C}$, we consider the following two cases:

For the case $(\mu+1)\delta>h+\delta$, we are going to prove that the code $\mathcal{C}$ is capable of recovering any erasure pattern $\mathcal{E}=\{E_i\,:\,1\leqslant i\leqslant\ell+2\}$ with $\sum_{1\leqslant i\leqslant\ell+2}|E_i|\leqslant h+\delta-1$, where $E_i\subseteq A_i$ for $1\leqslant i\leqslant\ell+1$ and $E_{\ell+2}\subseteq S$. Note that the $(r,\delta)_i$-locality means that we only need to consider the case that $|E_{i_t}|\geqslant\delta$ for $1\leqslant i_t\leqslant\ell+1$ and $1\leqslant t\leqslant w$. Again by Theorem 1, (10), and the fact that

$$|E_{\ell+2}|+\sum_{1\leqslant t\leqslant w}|E_{i_t}|\leqslant\sum_{1\leqslant i\leqslant\ell+2}|E_i|\leqslant h+\delta-1,$$

we have that $\mathcal{E}$ is recoverable.

For the case $h+\delta\geqslant(\mu+1)\delta$, similarly, we prove that the code $\mathcal{C}$ is capable of recovering any erasure pattern $\mathcal{E}=\{E_i\;:\;1\leqslant i\leqslant\ell+2\}$ with $\sum_{1\leqslant i\leqslant\ell+1}|E_i|+|E_{\ell+2}|\leqslant$

$(\mu+1)\delta-1$, where $E_i \subseteq A_i$ for $1 \leqslant i \leqslant \ell+1$ and $E_{\ell+2} \subseteq S$. Similarly, we conclude that there are at most $w \leqslant \mu$ sets $E_{i_1}, E_{i_2}, \cdots, E_{i_w}$ with $|E_{i_t}| \geqslant \delta$ and $1 \leqslant i_t \leqslant \ell+1$ for $1 \leqslant t \leqslant w$. Note that $\sum_{1 \leqslant i \leqslant \ell+1} |E_i| + |E_{\ell+2}| \leqslant (\mu+1)\delta - 1$ implies that $w \leqslant \mu$. By (10), we may conclude that

$$\left| A_{i_t} \cap \left(\bigcup_{t \neq j \in [w]} A_{i_j} \right) \right| \leqslant \delta - 1.$$

Now the fact that $\sum_{1 \leqslant t \leqslant w} |E_{i_t}| + |E_{\ell+2}| \leqslant \sum_{1 \leqslant i \leqslant \ell+1} |E_i| + |E_{\ell+2}| \leqslant (\mu+1)\delta - 1 \leqslant h + \delta - 1$ implies that $\mathcal{E}$ is recoverable by Theorem 1.

Finally, the optimality of $\mathcal{C}$ follows directly from Lemma 1 and the fact that $h + \delta \leqslant (\mu+1)\delta$. ■

A. Optimal Locally Repairable Codes With $(r, \delta)_i$-Locality Based on Packings or Steiner Systems

Based on Corollary 1, to construct optimal locally repairable codes we only need to find $\mathcal{A}$ such that (10) holds. In this section, we consider the case that $\mathcal{A}$ forms a combinatorial structure which satisfies the condition given by (10). We first consider a condition on the intersection of any pair of sets in $\mathcal{A}$ rather than μ sets as in (10).

Theorem 2: Assume the setting of Construction A. Let $\mathcal{A}$ be a set system formed by subsets of $\mathbb{F}_q \setminus S$, where S is an h-subset of $\mathbb{F}_q$. If there exists a positive integer a such that $|A_i \cap A_j| \leqslant a$ for all $i \neq j$, then the code $\mathcal{C}$ generated by Construction A is an $[n, k, d \geqslant \min\{h + \delta, (\lceil \frac{\delta}{a} \rceil + 1)\delta\}]_q$ linear code with $(r, \delta)_i$-locality. If additionally, $h \leqslant \lceil \frac{\delta}{a} \rceil \delta$, then the code $\mathcal{C}$ generated by Construction A is an optimal $[n, k, d = h + \delta]_q$ linear code with $(r, \delta)_i$-locality.

Proof: Let $\mu = \lceil \frac{\delta}{a} \rceil$. Then for any μ-subset, $\mathcal{R} \subseteq \mathcal{A}$, and for any $A' \in \mathcal{R}$, we have

$$\left| A' \cap \left(\bigcup_{A \in \mathcal{R} \setminus \{A'\}} A \right) \right| \leqslant (\mu - 1)a = \left(\left\lceil \frac{\delta}{a} \right\rceil - 1 \right) a \leqslant \delta - 1,$$

since $|A_i \cap A_j| \leqslant a$. The first claim follows from Corollary 1. Note that $\mu\delta \geqslant \lceil \frac{\delta}{a} \rceil \delta \geqslant h$ means that $(\mu+1)\delta \geqslant h + \delta$. Again by Corollary 1 we have the desired result follows. ■

Based on Theorem 2, we can use combinatorial designs to generate optimal locally repairable codes via Construction A. The following corollaries follow directly from Theorem 2.

Corollary 2: Let S be an h-subset of $\mathbb{F}_q$. If there exists a $(\tau+1)$-$(q-h, r+\delta-1, 1)$-packing $(\mathbb{F}_q \setminus S, \mathcal{B})$ and $0 \leqslant h \leqslant \lceil \frac{\delta}{\tau} \rceil \delta$, then there exists an optimal $[n, k, d]_q$ linear code with $(r, \delta)_i$-locality, where $n = |\mathcal{B}|(r + \delta - 1) + h - r + v$, $k = (|\mathcal{B}| - 1)r + v$, $0 < v \leqslant r$, and $d = h + \delta$.

Corollary 3: If there exists a $(\tau+1, r+\delta-1, q-h)$-Steiner system and $0 \leqslant h \leqslant \lceil \frac{\delta}{\tau} \rceil \delta$, then there exists an optimal $[n, k, d]_q$ linear code with $(r, \delta)_i$-locality, where

$$n = \frac{\binom{q-h}{\tau+1}(r+\delta-1)}{\binom{r+\delta-1}{\tau+1}} + h + v - r,$$

$$k = \left(\frac{\binom{q-h}{\tau+1}}{\binom{r+\delta-1}{\tau+1}} - 1 \right) r + v,$$

$0 < v \leqslant r$, and $d = h + \delta$.

B. Optimal Locally Repairable Codes With Order-Optimal Length: $(r, \delta)_i$-Locality

From a practical viewpoint, codes over small fields are preferred. This is due to the fact that smaller fields have much cheaper and faster implementations both in hardware and in software. Therefore, in coding theory, a common question is, given a desirable code length, what is the minimal required field size? Equivalently, we ask, given a field size, what is the maximal code length? In particular, for locally repairable codes, finding the maximal length of optimal locally repairable codes with $(r, \delta)_a$-locality was the subject of [22] and [11], for the cases of $\delta = 2$ and $\delta \geqslant 2$, respectively. Both constructions and bounds are proposed there. It is therefore natural to further ask how long can optimal locally repairable codes with $(r, \delta)_i$-locality be. This question is also important to us in order to analyze the performance of Construction A.

Theorem 3: Let $n = k + \ell(\delta - 1) + h$, $\delta \geqslant 2$, $k = \ell r$. Assume there exists an optimal $[n, k, d]_q$ linear code $\mathcal{C}$ with $(r, \delta)_i$-locality. For any given integer $0 \leqslant a \leqslant h$ define $T(a) = \lfloor (d - a - 1)/\delta \rfloor$. If $T(a) \geqslant 2$, then

$$n \leqslant \begin{cases} \frac{r+\delta-1}{r} \left(\frac{T(a)-1}{2(q-1)} q^{\frac{2(h-a-1)}{T(a)-1}} + a + 1 \right) - \frac{h(\delta-1)}{r}, & \text{if } T(a) \text{ is odd}, \\ \frac{r+\delta-1}{r} \left(\frac{T(a)}{2(q-1)} q^{\frac{2(h-a)}{T(a)}} + a \right) - \frac{h(\delta-1)}{r}, & \text{if } T(a) \text{ is even}, \end{cases}$$

where h can be rewritten as $h = d - \delta$.

The technical proof and its supporting lemmas are included in Appendix A.

Throughout the paper we shall look at the asymptotics of families of codes with locality. In the terminology of Theorem 3 we assume r, δ, h, d (and therefore a) are all constants. If the codes we study are optimal (with respect to the bound of Lemma 1), then k may be derived from n. Thus, we are left with the asymptotics of n as a function of the field size q. Therefore, we shall say the family of codes is *order optimal* if, up to a constant factor, it attains the bound of Theorem 3, namely,

$$n = \begin{cases} \Theta\left(q^{\frac{2(h-a-1)}{T(a)-1} - 1} \right) & \text{if } T(a) \text{ is odd}, \\ \Theta\left(q^{\frac{2(h-a)}{T(a)} - 1} \right) & \text{if } T(a) \text{ is even}. \end{cases}$$

Now, based on Theorem 3, we can analyze the performance of Construction A. The number of blocks of a packing is upper bounded by the following Johnson bound [26]:

Lemma 4 ([26]): The maximum possible number of blocks of a $(\tau+1)$-$(n_1, r+\delta-1, 1)$-packing $(X, \mathcal{B})$ is bounded by

$$|\mathcal{B}| \leqslant \left\lfloor \frac{n_1}{r+\delta-1} \left\lfloor \frac{n_1 - 1}{r+\delta-2} \left\lfloor \cdots \left\lfloor \frac{n_1 - \tau}{r+\delta-1-\tau} \right\rfloor \cdots \right\rfloor \right\rfloor \right\rfloor.$$

Thus, the number of blocks for a $(\tau+1)$-$(n_1, r+\delta-1, 1)$-packing can be as large as $O(n_1^{\tau+1})$, when τ, r, and δ are regarded as constants.

Corollary 4: Let $n_1 = q - h$. If there exists a $(\tau+1)$-$(n_1, r+\delta-1, 1)$-packing with blocks $\mathcal{B}$, $|\mathcal{B}| = \Omega(n_1^{\tau+1})$, and $0 \leqslant h \leqslant \lceil \frac{\delta}{\tau} \rceil \delta$, then there exists an optimal $[n, k, d]_q$ linear code with $(r, \delta)_i$-locality, where $n = |\mathcal{B}|(r+\delta-1)+h+v-r =$

$\Omega(q^{\tau+1})$, $k = (|\mathcal{B}|-1)r + v$ and $d = h + \delta$. Furthermore, if $h \geqslant \delta+1$, $v = r$, and $\tau = \delta - 1$ the code based on the $(\tau+1)$-$(n_1, r+\delta-1, 1)$-packing has order-optimal length, where r, h, and δ are regarded as constants.

Proof: By Corollary 2, we have $n = |\mathcal{B}|(r+\delta-1)+h+v-r = \Omega(q^{\tau+1})$ for the code generated by Construction A. In Theorem 3, setting $a = h-\delta-1$, we have $T(a) = \lfloor \frac{d-1-a}{\delta} \rfloor = \lfloor \frac{h+\delta-a-1}{\delta} \rfloor = 2$. Therefore, for the case $v = r$, by Theorem 3 again

$$\begin{aligned} n &\leqslant \frac{r+\delta-1}{r}\left(\frac{T(a)}{2(q-1)}q^{\frac{2(h-a)}{T(a)}} + a\right) - \frac{h(\delta-1)}{r} \\ &= \frac{r+\delta-1}{r}\left(\frac{1}{q-1}q^{\delta+1} + a\right) - \frac{h(\delta-1)}{r} = O(q^{\delta}). \end{aligned}$$

Thus, for the case $\tau = \delta - 1$ and $v = r$, the code $\mathcal{C}$ has length $n = \Omega(q^{\tau+1}) = \Omega(q^{\delta})$, which is order optimal with respect to the bound in Theorem 3, when h, r, and δ are regarded as constants. ■

Corollary 5: Let $n_1 = q - h$. If there exists a $(\tau+1, r+\delta-1, n_1)$-Steiner system and $0 \leqslant h \leqslant \lceil \frac{\delta}{\tau} \rceil \delta$, then there exists an optimal $[n,k,d]_q$ linear code with $(r,\delta)_i$-locality, where

$$\begin{aligned} n &= \frac{\binom{q-h}{\tau+1}(r+\delta-1)}{\binom{r+\delta-1}{\tau+1}} + h, \\ k &= \left(\frac{\binom{q-h}{\tau+1}}{\binom{r+\delta-1}{\tau+1}} - 1\right) r, \end{aligned}$$

and $d = h + \delta$. In particular, for the case $h \geqslant \delta + 1$ and $\tau = \delta - 1$, the code based on the $(\delta, r+\delta-1, q-h)$-Steiner system has order-optimal length, where h, r, and δ are regarded as constants.

Proof: The first part of the corollary follows directly from Corollary 3 and Definition 6. For the second part, the fact $h \geqslant \delta+1$ means that we can set $a = h - \delta - 1$ and $T(a) = \lfloor \frac{d-1-a}{\delta} \rfloor = \lfloor \frac{h+\delta-a-1}{\delta} \rfloor = 2$ in Theorem 3, which also means the code $\mathcal{C}$ has length

$$\begin{aligned} n &\leqslant \frac{r+\delta-1}{r}\left(\frac{T(a)}{2(q-1)}q^{\frac{2(h-a)}{T(a)}} + a\right) - \frac{h(\delta-1)}{r} \\ &= \frac{r+\delta-1}{r}\left(\frac{1}{q-1}q^{\delta+1} + a\right) - \frac{h(\delta-1)}{r} = O(q^{\delta}), \end{aligned}$$

for $v = r$. Now the conclusion comes from the fact that the upper bound is $O(q^{\delta})$ and the constructed code has length $n = \frac{\binom{n_1}{\tau+1}(r+\delta-1)}{\binom{r+\delta-1}{\tau+1}} + h = \Omega(q^{\delta})$, where we assume h, r, and δ are constants. ■

Remark 3: For the existence of packings in general the reader may refer to [38] and the survey in [14, VI.40].

Remark 4: Given positive integers τ, r and $\delta > 2$, the natural necessary conditions for the existence of a $(\tau+1, r+\delta-1, t-r+v)$-Steiner system are that $\binom{r+\delta-1-i}{\tau+1-i} | \binom{t-r+v-i}{\tau+1-i}$ for all $0 \leqslant i \leqslant \tau$. It was shown in [27] that these conditions are also sufficient except perhaps for finitely many cases. While q might not be a prime power, any prime power $\overline{q} \geqslant q$ will suffice for our needs. It is known, for example, that there is always a prime in the interval $[q, q+q^{21/40}]$ (see [1]). Thus, by Corollary 5, for all large enough t, there exists an optimal $[n,k,d]_q$ locally repairable code, with $(r,\delta)_i$-locality, where q is a prime power with $t \leqslant q \leqslant t + t^{21/40}$ and

$$\begin{aligned} n &= (r+\delta-1)\cdot\frac{\binom{t-r+v}{\tau+1}}{\binom{r+\delta-1}{\tau+1}} + h = \Omega(t^{\tau+1}) = \Omega(q^{\tau+1}), \\ k &= \frac{r\binom{t-r+v}{\tau+1}}{\binom{r+\delta-1}{\tau+1}}, \\ d &= h + \delta. \end{aligned}$$

Remark 5: One well known construction for optimal locally repairable codes with $(r,\delta)_i$-locality is that of pyramid codes. The pyramid code is based on an MDS code whose length is upper bounded by $q+d-2$ (and by the MDS conjecture this may be reduced to $q+1$ for q odd [2]). Thus, the length of pyramid code is upper bounded by $q+d-1-\delta+\lceil \frac{k}{r} \rceil(\delta-1) \leqslant q+d-1-\delta+\lceil \frac{q-1}{r} \rceil(\delta-1)$ (we note that $q+2-\delta+\lceil \frac{k}{r} \rceil(\delta-1) \leqslant q+2-\delta+\lceil \frac{q-d+2}{r} \rceil(\delta-1)$ according to MDS conjecture for the case of q odd), where $d \geqslant \delta$. According to our construction and bound (in Theorem 3), it follows that the pyramid code is sub-optimal in terms of asymptotic length, since we construct locally repairable codes with $(r,\delta)_i$-locality and length $n = \Omega(q^{\delta})$. In [43], also based on polynomial evaluations, locally repairable codes known as Tamo-Barg codes with $(r,\delta)_a$-locality and length $n = \Omega(q)$ are constructed. Compared with Tamo-Barg codes, our codes may have super-linear length in the field size, but only have $(r,\delta)_i$-locality.

Example 2: Set $n = 24$, $k = 14$, $\delta = 2$, $r = 2$, and $h = 3$. Let $\mathcal{A} = \{A_i \ : \ A_i \triangleq \{3,6,5\} + i \pmod 7, i \in \{0,1,\cdots,6\}\}$. According to Construction A, we can construct a linear code $\mathcal{C}$ with $(2,2)_i$-locality over $\mathbb{F}_{11}$, whose parity check matrix can be given as (transposed form):

$$H^{\mathsf{T}} = \begin{pmatrix} 7&0&0&0&0&0&0&5&1&2\\ 5&0&0&0&0&0&0&10&4&6\\ 0&7&0&0&0&0&0&2&8&10\\ 0&5&0&0&0&0&0&7&5&7\\ 0&0&7&0&0&0&0&7&1&3\\ 0&0&5&0&0&0&0&1&9&6\\ 0&0&0&3&0&0&0&1&3&7\\ 0&0&0&9&0&0&0&6&3&3\\ 0&0&0&0&3&0&0&3&6&8\\ 0&0&0&0&9&0&0&5&3&8\\ 0&0&0&0&0&9&0&1&4&7\\ 0&0&0&0&0&3&0&3&2&2\\ 0&0&0&0&0&0&7&2&9&4\\ 0&0&0&0&0&0&5&8&5&2\\ 10&0&0&0&0&0&0&0&0&0\\ 0&10&0&0&0&0&0&0&0&0\\ 0&0&10&0&0&0&0&0&0&0\\ 0&0&0&10&0&0&0&0&0&0\\ 0&0&0&0&10&0&0&0&0&0\\ 0&0&0&0&0&10&0&0&0&0\\ 0&0&0&0&0&0&10&0&0&0\\ 0&0&0&0&0&0&0&10&0&0\\ 0&0&0&0&0&0&0&0&10&0\\ 0&0&0&0&0&0&0&0&0&10 \end{pmatrix}.$$

Verified by a computer program, the minimum Hamming distance of $\mathcal{C}$ is 5. Thus, in this setting, Construction A generates a $[24, 14, 5]_{11}$ optimal linear code with $(2, 2)_i$-locality, consistent with the result in Theorem 2. Note that, to construct a code sharing the same parameters via the pyramid code, we need an MDS code with parameters $[18, 14, 5]_q$. However, according to the MDS conjecture this MDS code exists only under the condition that $q \geqslant 17$. Without the help of MDS conjecture, based on the result proposed in [2], we have $q \geqslant 16$ for this special setting.

Remark 6: For the case $\delta = 2$ and $d = 5$, optimal linear codes with all symbol $(r, 2)$-locality and order-optimal length $\Theta(q^2)$ have been introduced in [3], [22], [25]. The constructions in [3], [25] are given by parity-check matrices with 3 or 4 global parity checks, which means they only work for the cases $d = 5, 6$. One can verify that our construction still works for more general cases even if we restrict to the case $\delta = 2$.

Remark 7: For the case $\delta \geqslant 2$ and $d = 2\delta+1$, optimal linear codes with all symbol $(r, 2)$-locality and order-optimal length $\Theta(q^\delta)$ have been introduced in [11]. However, the construction in [11] should satisfy the condition $h \leqslant r + \delta - 1$, which is not needed for Construction A.

IV. Generalized Sector-Disk Codes

By Theorem 1, we may have extra benefits if $\left|\bigcup_{|E_i|\geqslant\delta, i\in[\ell+1]} E_i\right| < \sum_{|E_i|\geqslant\delta, i\in[\ell+1]} |E_i|$. In this section, we are going to use this property to construct array codes that can recover from special erasure patterns beyond the minimum Hamming distance. The basic idea of those constructions is to let all the code symbols share the same evaluation point in step 1 of Construction A in the same column of an array code. Then for this array code, one erased column may only increase the value $\left|\bigcup_{|E_i|\geqslant\delta, i\in[\ell+1]} E_i\right|$ by one. Hence, when we consider sector-disk-like erasure patterns, we may get some extra benefit beyond the minimum Hamming distance. We begin with some definitions.

Definition 7: Let $\mathcal{C}$ be an optimal $[n, k, d]_q$ linear code with $(r, \delta)_i$-locality. Then the code $\mathcal{C}$ is said to be an (γ, s)-*generalized sector-disk code* (GSD code) if the codewords can be arranged into an array

$$C = \begin{pmatrix} c_{1,1} & c_{1,2} & \cdots & c_{1,a} \\ c_{2,1} & c_{2,2} & \cdots & c_{2,a} \\ \vdots & \vdots & \ddots & \vdots \\ c_{b,1} & c_{b,2} & \cdots & c_{b,a} \end{pmatrix}$$

such that:

(I) all the erasure patterns that contain any γ columns and additional s sectors can be recovered; and

(II) $\gamma b + s > d - 1$.

Remark 8: In the definition of GSD codes, the condition $\gamma b + s > d - 1$ (Condition II) means that we are considering codes that can repair more (sector) erasures than the minimum distance-based threshold.

Remark 9: If the code $\mathcal{C}$ has $(r, \delta)_a$-locality, the repair sets are exactly the rows, and then the $(\delta - 1, d - \delta)$-GSD code is exactly the $(\delta - 1, d - \delta)$-SD code [34], where we use the fact that a $(\delta - 1, s)$-SD code is always an optimal locally repairable code with $d = s + \delta$. Compared with SD codes, GSD codes relax the conditions in the following three aspects:

- GSD codes only require $(r, \delta)_i$-locality, whereas SD codes require $(r, \delta)_a$-locality;
- A row in an array codeword of a GSD code is not necessary a repair set;
- The number of column erasures is not restricted to $\delta - 1$ as in SD codes.

In the following construction, we use Construction A to generate GSD codes.

Construction B: Let S be an h-subset of $\mathbb{F}_q$ and let $(X = \mathbb{F}_q \setminus S, \mathcal{A} = \{A_i : 1 \leqslant i \leqslant \ell + 1\})$ be a t-regular $(m = q - h, r + \delta - 1, 1)$-packing, where $A_i = \{\theta_{i,j} : 1 \leqslant j \leqslant r + \delta - 1\}$ for $1 \leqslant i \leqslant \ell + 1$. Based on $\mathcal{A}$ and S, we can generate a locally repairable code $\mathcal{C}$ according to Construction A. Define column vectors $V_\tau \in \mathbb{F}_q^t$ for $\tau \in X$ as

$$V_\tau^\intercal = (c_{i_{\tau,1}, j_{\tau,1}}, c_{i_{\tau,2}, j_{\tau,2}}, \ldots, c_{i_{\tau,t}, j_{\tau,t}}),$$

where

$$\theta_{i_{\tau,b}, j_{\tau,b}} = \tau, \text{ for } 1 \leqslant b \leqslant t.$$

Herein, we highlight that we have m columns of a codeword that are given by $V_\tau^\intercal \in \mathbb{F}_q^t$ for $\tau \in X$. Arrange the h global parity symbols as the last $\lceil \frac{h}{t} \rceil$ columns. If there are empty sectors in the array, then we fill them with 0.

Theorem 4: Let $\mathcal{C}$ be the $t \times (m + \lceil \frac{h}{t} \rceil)$ array code generated by Construction B. Then each element of the first m columns has (r, δ)-locality. If $h \leqslant \delta^2$, then the code can recover from any $h + \delta - 1$ erasures. Furthermore:

(I) The code $\mathcal{C}$ can recover from any erasure pattern of $y \leqslant 2$ columns from the first m columns and any other $h - y - 1$ erasures.

(II) If $\binom{y}{2} \leqslant \delta$, then the code $\mathcal{C}$ can recover from any erasure pattern of y columns from the first m columns and any other $h - 2 - \binom{y}{2}$ erasures.

(III) The code $\mathcal{C}$ can recover from any erasure pattern of $y < \frac{(\delta+1)\delta}{2} - 1$ columns from the first m columns and any other $\min\{\frac{(\delta+1)\delta}{2} - y - 1, h + \delta - 1 - y\}$ erasures.

Proof: By Lemma 3 and Theorem 1, we only need to prove that the desired erasure patterns satisfy (3) and (4). Since $\mathcal{A}$ forms a t-regular $(m, r+\delta-1, 1)$-packing, for the condition given by (4) we consider a sufficient condition that is the erasure pattern contains at most δ repair sets with each of them containing more than δ erasures.

For case (I) and $y = 2$, say the erased columns are marked by θ_1 and θ_2. We focus on the repair sets with more than δ erasures. In those repair sets, there is at most one repair set that contains θ_1 and θ_2, while the remaining repair sets contain at most one of them. For this case, we need at least $\delta - 2 + (\delta - 1)(\delta - 1) + \delta - 2 = \delta^2 - 3 \geqslant h - 3$ erasures before we achieve $\delta+1$ repair sets with each of them containing more than δ erasures. Thus, the code $\mathcal{C}$ can recover from any erasure pattern of $y=2$ columns from the first m columns and any other $h-3$ erasures. The same analysis proves the case $y=1$.

For the case (II), we assume that the erased columns are marked by elements in $\Theta=\{\theta_1,\theta_2,\cdots,\theta_y\}$. Note that $\mathcal{A}$ is an $(m,r+\delta-1,1)$-packing, which means that each 2-subset of Θ appears in at most one repair set. This is to say, for any δ repair sets $A_{j_1},A_{j_2},\cdots,A_{j_\delta}$ we have

$$\sum_{1\leqslant i\leqslant\delta}|\Theta\cap A_{j_i}|\leqslant 2\binom{y}{2}+\delta-\binom{y}{2}=\binom{y}{2}+\delta,$$

which means for any $E_{j_i}\subseteq A_{j_i}$

$$\sum_{1\leqslant i\leqslant\delta}|\Theta\cap E_{j_i}|\leqslant 2\binom{y}{2}+\delta-\binom{y}{2}=\binom{y}{2}+\delta.$$

Therefore, for this case, we need at least $\delta^2+\delta-2-\binom{y}{2}-\delta\geqslant h-2-\binom{y}{2}$ erasures before we achieve $\delta+1$ repair sets each of which contains more than δ erasures. In other words, the code $\mathcal{C}$ can recover from any erasure pattern of y columns from the first m columns and any other $h-2-\binom{y}{2}$ erasures.

For the case (III), we assume that the erased columns are marked by elements in $\Theta=\{\theta_1,\theta_2,\cdots,\theta_y\}$. Note that $\mathcal{A}$ is an $(m,w,1)$-packing, which means that $A_i\cap A_j\leqslant 1$ for $1\leqslant i,j\leqslant m$ with $i\neq j$. For any $\delta+1$ repair sets $A_{j_1},A_{j_2},\cdots,A_{j_{\delta+1}}$ we have

$$\left|\left(\bigcup_{1\leqslant i\leqslant\delta+1}A_{j_i}\right)\right|\geqslant\sum_{1\leqslant i\leqslant\delta+1}|A_{j_i}|-\binom{\delta+1}{2}.$$

Thus, for any $E_{j_i}\subseteq A_{j_i}$ with $|E_{j_i}|\geqslant\delta$ for $1\leqslant i\leqslant\delta+1$, we have

$$\begin{aligned}\left|\left(\bigcup_{1\leqslant i\leqslant\delta+1}E_{j_i}\right)\right|&\geqslant\sum_{1\leqslant i\leqslant\delta+1}|E_{j_i}|-\binom{\delta+1}{2}\\&\geqslant(\delta+1)\delta-\binom{\delta+1}{2},\end{aligned}$$

which means that we need at least $\binom{\delta+1}{2}-y-1$ erasures before we achieve $\delta+1$ repair sets each of which contains more than $\delta+1$ erasures. Thus, the code $\mathcal{C}$ can recover from any erasure pattern of y columns from the first m columns and any other $\binom{\delta+1}{2}-y-1$ erasures if $h+\delta-1-y\geqslant\binom{\delta+1}{2}-y-1>0$. ■

Example 3: Set $n=24$, $k=14$, $\delta=2$, $r=2$, and $h=3$. Let $\mathcal{A}=\{A_i\ :\ A_i\triangleq\{3,6,5\}+i\subseteq\mathbb{Z}_7,i\in\mathbb{Z}_7\}$. According to Construction B, we can modify the code in Example 2 into a 3×8 array code, whose parity-check matrix can be given as (transposed form):

$$H^{\mathsf{T}}=\left(\begin{array}{cccccccccc}0&0&0&9&0&0&0&6&3&3\\0&0&0&0&0&9&0&1&4&7\\0&0&0&0&0&0&10&0&0&0\\\hline 10&0&0&0&0&0&0&0&0&0\\0&0&0&0&9&0&0&5&3&8\\0&0&0&0&0&0&7&2&9&4\\\hline 7&0&0&0&0&0&0&5&1&2\\0&10&0&0&0&0&0&0&0&0\\0&0&0&0&0&3&0&3&2&2\\\hline 0&7&0&0&0&0&0&2&8&10\\0&0&10&0&0&0&0&0&0&0\\0&0&0&0&0&0&5&8&5&2\\\hline 5&0&0&0&0&0&0&10&4&6\\0&0&7&0&0&0&0&7&1&3\\0&0&0&10&0&0&0&0&0&0\\\hline 0&5&0&0&0&0&0&7&5&7\\0&0&0&3&0&0&0&1&3&7\\0&0&0&0&10&0&0&0&0&0\\\hline 0&0&5&0&0&0&0&1&9&6\\0&0&0&0&3&0&0&3&6&8\\0&0&0&0&0&10&0&0&0&0\\\hline 0&0&0&0&0&0&0&10&0&0\\0&0&0&0&0&0&0&0&10&0\\0&0&0&0&0&0&0&0&0&10\end{array}\right).$$

Verified by a computer program, the array code can recover from any 2 column erasures from the first 7 columns, which is consistent with the result in Theorem 4. Note that this kind of erasure pattern is beyond the minimum Hamming distance $d=5$ as shown in Example 2.

By Construction B, we can generate codes that may recover from some special erasure patterns beyond the minimum Hamming distance. However, all those erasure patterns do not treat columns equally, and distinguish between two types of columns. If this is an unwanted feature, we may arrange the global parity checks across columns, as done in the following construction.

Construction C: Let S be a h-subset of $\mathbb{F}_q$ and let $(X\subseteq\mathbb{F}_q\setminus S,\mathcal{A}=\{A_i\ :\ 1\leqslant i\leqslant\ell+1\})$ be a t-regular $(m,r+\delta-1,1)$-packing, where $A_i=\{\theta_{i,j}\ :\ 1\leqslant j\leqslant r+\delta-1\}$ for $1\leqslant i\leqslant\ell+1$. Let $n=v|X|=v\rho$ with $v\geqslant t$, then based on $\mathcal{A}$ and S, we can generate a locally repairable code $\mathcal{C}$ according to Construction A. List the elements of X as $(x_1,x_2,\cdots,x_\rho)$. Define column vectors $V_{x_a}\in\mathbb{F}_q^v$ for $a\in[\rho]$ as

$$\begin{aligned}V_{x_a}^{\mathsf{T}}=&(c_{i_{x_a,1},j_{x_a,1}},c_{i_{x_a,2},j_{x_a,2}},\ldots,c_{i_{x_a,t},j_{x_a,t}},c_{\ell+2,(a-1)h/\rho+1},\\&c_{\ell+2,(a-1)h/\rho+2},\cdots,c_{\ell+2,ah/\rho}),\end{aligned}$$

where

$$\theta_{i_{x_a,b},j_{x_a,b}}=x_a,\ \text{for } 1\leqslant b\leqslant t. \tag{11}$$

Remark 10: In Construction C, the fact that $(X\subseteq\mathbb{F}_q\setminus S,\mathcal{A}=\{A_i\ :\ 1\leqslant i\leqslant\ell+1\})$ is a regular packing means that $n-h=t\rho$. Thus, by $n=v\rho$, we have $\rho\mid h$ and $v=t+h/\rho$. This is to say the array given by (11) is well defined.

Corollary 6: Let $\mathcal{C}$ be the $v\times\rho$ array code generated by Construction C. Then $\mathcal{C}$ has $(r,\delta)_i$-locality. If $h\leqslant\delta^2$, then the code can recover from any $h+\delta-1$ erasures. Furthermore:

(I) The code $\mathcal{C}$ can recover from any erasure pattern of $y \leqslant 2$ columns and any other $h - y(v - t + 1) - 1$ erasures.

(II) If $\binom{y}{2} \leqslant \delta$, then the code $\mathcal{C}$ can recover from any erasure pattern of y columns and any other $h-2-\binom{y}{2}-y(v-t)$ erasures.

(III) The code $\mathcal{C}$ can recover from any erasure pattern of $y < \frac{(\delta+1)\delta}{2} - 1$ columns and any other $\min\{\frac{(\delta+1)\delta}{2} - y(v - t + 1) - 1, h + \delta - 1 - y(v - t + 1)\}$ erasures.

Proof: Note that any y columns of $\mathcal{C}$ can be regarded as y columns from the first m columns and $y(v - t)$ erasures (sectors) from the global check symbols, for the code generated by Construction B. Thus, the desired results follows directly from Theorem 4, respectively. ■

For the case $r \nmid k$ and $h = r - v$, we may modify Construction C as follows.

Construction D: Let S be an $(r - v)$-subset of $\mathbb{F}_q$ and let $(X \subseteq \mathbb{F}_q \setminus S, \mathcal{B} = \{B_i \ : \ 1 \leqslant i \leqslant \ell + 1\})$ be a t-regular $(m, r + \delta - 1, 1)$-packing. Let $A_i = B_i$ for $1 \leqslant i \leqslant \ell$ and $A_{\ell+1} \subseteq B_{\ell+1}$. Let $n = t|X| = t\rho$ and $k = \ell r + v$, then based on $\mathcal{A}$ and S, we can generate a locally repairable code $\mathcal{C}$ according to Construction A. List the elements of $B_{\ell+1} \setminus A_{\ell+1}$ as $(x_1, x_2, \ldots, x_{r-v})$ and X as $(x_1, x_2, \cdots, x_\rho)$. Define column vectors $V_{x_a} \in \mathbb{F}_q^v$ for $a \in [\rho]$ as

$$V_{x_a}^{\mathsf{T}} = \begin{cases} (c_{i_{x_a,1},j_{x_a,1}}, c_{i_{x_a,2},j_{x_a,2}}, \ldots, c_{i_{x_a,t-1},j_{x_a,t-1}}, c_{\ell+2,a}), & \text{if } 1 \leqslant a \leqslant r - v, \\ (c_{i_{x_a,1},j_{x_a,1}}, c_{i_{x_a,2},j_{x_a,2}}, \ldots, c_{i_{x_a,t},j_{x_a,t}}), & \text{otherwise,} \end{cases}$$

where $\theta_{i_{x_a,b},j_{x_a,b}} = x_a$, $1 \leqslant b \leqslant t - 1$ for $1 \leqslant a \leqslant r - v$ and $1 \leqslant b \leqslant t$ for $r - v + 1 \leqslant a \leqslant \rho$.

Corollary 7: Let $\mathcal{C}$ be the $t \times \rho$ array code generated by Construction D. Then $\mathcal{C}$ has $(r, \delta)_i$-locality. If $h \leqslant \delta^2$, then the code can recover any $h + \delta - 1$ erasures. Furthermore:

(I) The code $\mathcal{C}$ can recover from any erasure pattern of $y \leqslant 2$ columns and any other $h - 2y - 1$ erasures.

(II) If $\binom{y}{2} \leqslant \delta$, then the code $\mathcal{C}$ can recover from any erasure pattern of y columns and any other $h - 2 - \binom{y}{2} - y$ erasures.

(III) The code $\mathcal{C}$ can recover from any erasure pattern of $y < \frac{(\delta+1)\delta}{2} - 1$ columns and any other $\min\{\frac{(\delta+1)\delta}{2} - 2y - 1, h + \delta - 1 - 2y\}$ erasures.

Proof: Note that any y columns of $\mathcal{C}$ can be regarded as y columns from the first m columns and at most y erasures (sectors) from the global check symbols, for the code generated by Construction B. Thus, the desired results follows directly from Theorem 4, respectively. ■

Based on known results about regular packings, we derive some parameters of GSD codes resulting from our constructions. In particular, we use two well known classes of Steiner systems that are the affine geometries and projective geometries.

Lemma 5 ([14]): Let $\beta \geqslant 2$ be an integer and q_1 a prime power, then there exists a $(2, q_1, q_1^\beta)$-Steiner system.

Based on affine geometries and Construction D, we have the following conclusion for GSD codes.

Corollary 8: Let $\beta \geqslant 2$ be an integer and q_1 a prime power. Set $q_1 = r + \delta - 1$, $n = \frac{q_1^\beta(q_1^\beta-1)}{q_1-1}$, $\delta \geqslant 2$, $k = (\frac{q_1^{\beta-1}(q_1^\beta-1)}{q_1-1} - 1)r + v$ with $1 \leqslant v \leqslant r - 1$, and $h = r - v = q_1 - \delta - v + 1$. Let $\mathcal{C}$ be the $\frac{q_1^\beta-1}{q_1-1} \times q_1^\beta$ array code generated by Construction D using a $(2, q_1, q_1^\beta)$-Steiner system from Lemma 5. If $h \leqslant \delta^2$, then the code $\mathcal{C}$ is an $[n, k, h + \delta - 1]_q$ optimal locally repairable code with $(r, \delta)_i$-locality, where $q \geqslant q_1^\beta + h$. Furthermore:

(I) If $y \leqslant 2$ and $y(\frac{q_1^\beta-1}{q_1-1} - 2) > \delta$, then the code $\mathcal{C}$ is a $(y, h - 2y - 1)$-GSD code.

(II) If $\binom{y}{2} \leqslant \delta$ and $y\frac{q_1^\beta-1}{q_1-1} - 1 - \binom{y}{2} - y > \delta$, then the code $\mathcal{C}$ is a $(y, h - 2 - \binom{y}{2} - y)$-GSD code.

(III) If $y < \frac{(\delta+1)\delta}{2} - 1$ and $y\frac{q_1^\beta-1}{q_1-1} + s > h + \delta - 1$, then the code $\mathcal{C}$ is a (y, s)-GSD code, where $s = \min\{\frac{(\delta+1)\delta}{2} - 2y - 1, h + \delta - 1 - 2y\}$ erasures.

Herein, we highlight that the second restriction of each item comes from the requirement in Definition 7-(II).

Proof: The proof follows directly from Corollary 7, Lemma 5, and Definition 7. ■

Lemma 6 ([14]): Let $\beta \geqslant 2$ be an integer and q_1 a prime power, then there exists a $(2, q_1 + 1, \frac{q_1^{\beta+1}-1}{q_1-1})$-Steiner system.

Based on projective geometries and Construction D, we have the following conclusion for GSD codes.

Corollary 9: Let $\beta \geqslant 2$ be an integer and q_1 a prime power. Set $q_1 + 1 = r + \delta - 1$, $n = \frac{(q_1^{\beta+1}-1)(q_1^\beta-1)}{(q_1-1)^2}$, $\delta \geqslant 2$, $k = (\frac{(q_1^{\beta+1}-1)(q_1^\beta-1)}{(q_1-1)(q_1^2-1)} - 1)r + v$ with $1 \leqslant v \leqslant r - 1$, and $h = r - v = q_1 - \delta - v + 2$. Let $\mathcal{C}$ be the $\frac{q_1^\beta-1}{q_1-1} \times \frac{q_1^{\beta+1}-1}{q_1-1}$ array code generated by Construction D using a $(2, q_1 + 1, \frac{q_1^{\beta+1}-1}{q_1-1})$-Steiner system from Lemma 5. If $h \leqslant \delta^2$, then the code $\mathcal{C}$ is an $[n, k, h + \delta - 1]_q$ optimal locally repairable code with $(r, \delta)_i$-locality, where $q \geqslant \frac{q_1^{\beta+1}-1}{q_1-1} + h$ is a prime power. Furthermore:

(I) If $y \leqslant 2$ and $y(\frac{q_1^\beta-1}{q_1-1} - 2) > \delta$, then the code $\mathcal{C}$ is a $(y, h - 2y - 1)$-GSD code.

(II) If $\binom{y}{2} \leqslant \delta$ and $y\frac{q_1^\beta-1}{q_1-1} - 1 - \binom{y}{2} - y > \delta$, then the code $\mathcal{C}$ is a $(y, h - 2 - \binom{y}{2} - y)$-GSD code.

(III) If $y < \frac{(\delta+1)\delta}{2} - 1$ and $y\frac{q_1^\beta-1}{q_1-1} + s > h + \delta - 1$, then the code $\mathcal{C}$ is a (y, s)-GSD code, where $s = \min\{\frac{(\delta+1)\delta}{2} - 2y - 1, h + \delta - 1 - 2y\}$ erasures.

Proof: The proof follows directly from Corollary 7, Lemma 6, and Definition 7. ■

Lemma 7 ([14]): For any $\beta \geqslant 2$ and prime power q_1, there exists a $(3, q_1 + 1, q_1^\beta + 1)$-Steiner system.

Similarly, based on Steiner systems from spherical geometries and Construction D, we have the following conclusion for GSD codes.

Corollary 10: Let $\beta \geqslant 2$ be an integer and q_1 a prime power. Set $q_1 + 1 = r + \delta - 1$, $n = (q_1^\beta + 1)\frac{\binom{q_1^\beta}{2}}{\binom{q_1}{2}}$, $\delta \geqslant 2$, $k = (\frac{(q_1^\beta+1)\binom{q_1^\beta}{2}}{(q_1+1)\binom{q_1}{2}} - 1)r + v$ with $1 \leqslant v \leqslant r - 1$, and $h = r - v = q_1 - \delta - v + 2$. Let $\mathcal{C}$ be the $\frac{\binom{q_1^\beta}{2}}{\binom{q_1}{2}} \times (q_1^\beta + 1)$ array

TABLE I
A COMPARISON OF MR-CODES, SD-CODES, AND GSD-CODES

γ	s	Type	n	Ref.	Comment
any	1	MR	$\Theta(q)$	[5]	
$1, 2, 3$	2	SD	$\Theta(q)$	[34]	
any	2	SD	$\Theta(q)$	[7]	
any	3	MR	$\Theta(q^{1/3})$	[20]	
1	any	MR	$\Theta(q^{1/(s-1)})$	[18]	$r\|(k+s)$
any	any	MR	$\Theta(\log q)$	[12]	$\gamma = \delta - 1$
any	any	MR	$\Theta(q^{1/s})$	[17]	$\gamma = \delta - 1$
any	any	MR	$\Theta(q^{1/r})$	[33]	$\gamma = \delta - 1$
any	any	GSD	$\Theta(q^{2-1/\beta})$	Cor. 8,9	$\beta \geqslant 2$ is a constant
any	any	GSD	$\Theta(q^{3-2/\beta})$	Cor. 10	$\beta \geqslant 2$ is a constant
any	any	GSD	$\Theta(q^2)$	Cor. 8,9	q_1 is a constant
any	any	GSD	$\Theta(q^3)$	Cor. 10	q_1 is a constant
any	any	GSD	$\Theta(q^2)$	Cor. 11	e, u, h are constants
any	any	GSD	$\Theta(q^\tau)$	Cor. 7	Remark 11 (non-explicit)
any	any	GSD	$\Theta(q^\tau)$	Cor. 7	Remark 12 (non-explicit)

code generated by Construction D using a $(2, q_1+1, \frac{q_1^{\beta+1}-1}{q_1-1})$-Steiner system from Lemma 5. If $h \leqslant \delta^2$, then the code $\mathcal{C}$ is an $[n, k, h+\delta-1]_q$ optimal locally repairable code with $(r,\delta)_i$-locality, where $q \geqslant q_1^\beta + 1 + h$ is a prime power. Furthermore:

(I) If $y \leqslant 2$ and $y(\frac{\binom{q_1^\beta}{2}}{\binom{q_1}{2}} - 2) > \delta$, then the code $\mathcal{C}$ is a $(y, h-2y-1)$-GSD code.

(II) If $\binom{y}{2} \leqslant \delta$ and $y\frac{\binom{q_1^\beta}{2}}{\binom{q_1}{2}} - 1 - \binom{y}{2} - y > \delta$, then the code $\mathcal{C}$ is a $(y, h-2-\binom{y}{2}-y)$-GSD code.

(III) If $y < \frac{(\delta+1)\delta}{2} - 1$ and $y\frac{\binom{q_1^\beta}{2}}{\binom{q_1}{2}} + s > h + \delta - 1$, then the code $\mathcal{C}$ is a (y, s)-GSD code, where $s = \min\{\frac{(\delta+1)\delta}{2} - 2y - 1, h + \delta - 1 - 2y\}$ erasures.

Proof: The proof follows directly from Corollary 7, Lemma 7, and Definition 7. ∎

For regular packings, we have the following lemma due to a recursive construction from [26]. A direct construction is also supplied for the reader's convenience in Appendix B.

Lemma 8 ([26]): Let $n_2 = p_1^{m_1} p_2^{m_2} \cdots p_u^{m_u}$, where p_i are distinct primes, and $m_i > 0$, for all i. If $e | \gcd(p_1^{m_1}-1, p_2^{m_2}-1, \cdots, p_u^{m_u}-1)$, then there exists a $\frac{1}{e^u}\prod_{1 \leqslant i \leqslant u}(p_1^{m_i}-1)$-regular $(en_2, e, 1)$-packing.

Based on regular packings we can also generate GSD codes as follows.

Corollary 11: Let $n_2 = p_1^{m_1} p_2^{m_2} \cdots p_u^{m_u}$ and $e | \gcd(p_1^{m_1} - 1, p_2^{m_2}-1, \cdots, p_u^{m_u}-1)$, where p_i are distinct primes, and $m_i > 0$, for all i. Define $p = \prod_{1 \leqslant i \leqslant u}(p_1^{m_i}-1)$. Set $e = r + \delta - 1$, $n = n_2 p / e^{u-1}$, $\delta \geqslant 2$, $k = (n_2 p/e^u - 1)r + v$ with $1 \leqslant v \leqslant r-1$, and $h = r - v = e - \delta + 1 - v$. Let $\mathcal{C}$ be the $p/e^u \times en_2$ array code generated by Construction D. If $h \leqslant \delta^2$, then the code $\mathcal{C}$ is an $[n, k, h+\delta-1]_q$ optimal locally repairable code with $(r,\delta)_i$-locality, where $q \geqslant en_2 + h$ is a prime power. Furthermore:

(I) If $y \leqslant 2$ and $y(p/e^{u-1} - 2) > \delta$, then the code $\mathcal{C}$ is a $(y, h-2y-1)$-GSD code.

(II) If $\binom{y}{2} \leqslant \delta$ and $yp/e^{u-1} - 1 - \binom{y}{2} - y > \delta$, then the code $\mathcal{C}$ is a $(y, h-2-\binom{y}{2}-y)$-GSD code.

(III) If $y < \frac{(\delta+1)\delta}{2} - 1$ and $yp/e^{u-1} + s > h + \delta - 1$, then the code $\mathcal{C}$ is a (y, s)-GSD code, where $s = \min\{\frac{(\delta+1)\delta}{2} - 2y - 1, h + \delta - 1 - 2y\}$ erasures.

Table I lists some known results about SD codes and MR code (PMDS codes) as a comparison with the GSD codes we have constructed. The main point of comparison is the asymptotics of the length of the code with respect to the field size. For this table, $n = m(r+\delta-1)$ is the total number of sectors for a codeword, k is the number of sectors for information symbols, $r+\delta-1$ is the number of columns (i.e., the code has $(r,\delta)_a$-locality), and q is the field size. For a fair comparison with our results in Corollaries 8–11, we consider r, δ, and γ, as constants when we consider the relationship between n and q. We further make the following remarks:

Remark 11: By Corollaries 8 and 9, there exist GSD codes with $n = \Theta(q^2)$, where h, r, and δ are regarded as constants, i.e., q_1 is a constant, as already written in Table I. Note that if we regard $\beta \geqslant 2$ as a constant then $n = \Theta(q^{\frac{2\beta-1}{\beta}})$ with $q = \Theta(q_1^\beta)$. In addition, for general cases by using Steiner systems with parameters $(\tau, r+\delta-1, n_1)$, Steiner systems are capable of yielding optimal locally repairable codes (similarly, GSD codes) with length $n = \Theta(q^\tau)$ as shown in Corollary 5 and Remark 5. Here we apply the fact that Steiner systems are regular packings, which means that the locally repairable codes in Corollary 5 and Remark 5 can yield GSD codes by Construction D and Corollary 7. For example, in Corollary 10, we have $n = \Theta(q^3)$ for the case $\tau = 3$, where $q_1 + 1 = r + \delta - 1$ is regarded as a constant. However, the problem of constructing Steiner systems in general is widely open in combinatorics. For a summary of combinatorial designs and linear codes, the reader may refer to [15] for example.

Remark 12: According to Corollary 11, there exist GSD codes with $n = \frac{\prod_{1 \leqslant i \leqslant u}(p_i^{m_i}(p_i^{m_i}-1))}{e^u} = \Theta(q^2)$, where $q \geqslant e\prod_{1 \leqslant i \leqslant u} p_i^{m_i} + h$ and we consider $e = r+\delta-1$, u, and h as constants. In particular, for the case $\delta = 2$, this code has order-optimal length with respect to the bound in Theorem 3. Similarly, for $\tau > 2$, to generate codes with length $n = \Theta(q^\tau)$ we need regular τ-$(n_1, r+\delta-1, 1)$-packings with $\tau > 2$ (see Definition 5), where we also need to apply Construction D to rearrange the locally repairable codes into GSD codes.

Example 4: Set $n = 9 \times 73 = 657$, $k = 7 \times 72 + 1 = 505$, $\delta = 3$, $r = 7$, and $h = 6$. Let $\mathcal{A} = \{A_i \ : \ i \in [73]\}$ be a $(2, 9, 73)$-Steiner system. According to Construction D, we can generate a 9×73 array code, which forms a $(2, 1)$-GSD code (or a $(1, 3)$-GSD code). This code is an optimal $[657, 505, 9]_{q\geqslant 79}$ locally repairable code with $(7, 3)_i$-locality when viewed as a one dimensional linear code.

V. Locally Repairable Codes via Classical Goppa Codes

In this section, inspired by the classical Goppa code [21], we apply a similar method to construct locally repairable codes.

Construction E: Let $G_1(x)$ and $G_2(x)$ be two polynomials over $\mathbb{F}_{q^m}$ with degree $\delta - 1$ and h, respectively. Let $S = (\gamma_1, \gamma_2, \cdots, \gamma_n)$ be a sequence of length n over $\mathbb{F}_{q^m}$. Also, let $S_1, \ldots, S_{\ell+1} \subseteq \mathbb{F}_{q^m}$ be subsets such that $|S_i| = r + \delta - 1$ for $1 \leqslant i \leqslant \ell$, $|S_{\ell+1}| \leqslant h$, as well as,

$$\bigcup_{1\leqslant i\leqslant \ell+1} S_i = \{\gamma_i \ : \ 1 \leqslant i \leqslant n\}.$$

and

$$G_1(\gamma_i)G_2(\gamma_i) \neq 0 \quad \text{for } 1 \leqslant i \leqslant n.$$

Define the code $\Gamma_{q^m}(\mathcal{S} = \{S, S_1, \ldots, S_\ell\}, \mathcal{G} = \{G_1, G_2\})$ as a set of vectors $V = (v_1, v_2, \ldots, v_n) \in \mathbb{F}_{q^m}^n$ such that

$$\sum_{1\leqslant j\leqslant r+\delta-1} \frac{v_{i,j}}{x - \gamma_{i,j}} \equiv 0 \pmod{G_1(x)} \quad \text{for } 1 \leqslant i \leqslant \ell$$

and

$$\sum_{1\leqslant j\leqslant n} \frac{v_j}{x - \gamma_j} \equiv 0 \pmod{G_2(x)},$$

where for $1 \leqslant i \leqslant \ell$ and $1 \leqslant j \leqslant r + \delta - 1$, we denote $v_{(i-1)(r+\delta-1)+j}$ as $v_{i,j}$, and $\gamma_{(i-1)(r+\delta-1)+j}$ as $\gamma_{i,j}$, and $S_i = \{\gamma_{i,j} : 1 \leqslant j \leqslant r + \delta - 1\}$.

Lemma 9: The code $\Gamma_{q^m}(\mathcal{S}, \mathcal{G})$ generated by Construction E is an $[n, k]_{q^m}$ linear code with $k \geqslant n - \ell(\delta - 1) - h$, whose code symbol $v_{i,j}$ has (r, δ)-locality for $1 \leqslant i \leqslant \ell$ and $1 \leqslant j \leqslant r + \delta - 1$.

Proof: By the properties of classical Goppa codes (refer to [32] Chapter 12.3 for more details), the parity-check matrix of $\Gamma_{q^m}(\mathcal{S}, \mathcal{G})$ may be written as

$$P = \begin{pmatrix} P_{1,1} & 0 & \cdots & 0 & 0 \\ 0 & P_{1,2} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & P_{1,\ell} & 0 \\ P_{2,1} & P_{2,2} & \cdots & P_{2,\ell} & P_{2,\ell+1} \end{pmatrix},$$

where

$$P_{1,i} = \begin{pmatrix} G_1^{-1}(\gamma_{i,1}) & \cdots & G_1^{-1}(\gamma_{i,r+\delta-1}) \\ G_1^{-1}(\gamma_{i,1})\gamma_{i,1} & \cdots & G_1^{-1}(\gamma_{i,r+\delta-1})\gamma_{i,r+\delta-1} \\ \vdots & & \vdots \\ G_1^{-1}(\gamma_{i,1})\gamma_{i,1}^{\delta-3} & \cdots & G_1^{-1}(\gamma_{i,r+\delta-1})\gamma_{i,r+\delta-1}^{\delta-3} \\ G_1^{-1}(\gamma_{i,1})\gamma_{i,1}^{\delta-2} & \cdots & G_1^{-1}(\gamma_{i,r+\delta-1})\gamma_{i,r+\delta-1}^{\delta-2} \end{pmatrix}$$

for $1 \leqslant i \leqslant \ell$, and

$$\begin{aligned} &\begin{pmatrix} P_{2,1} & P_{2,2} & P_{2,3} & \cdots & P_{2,\ell} & P_{2,\ell+1} \end{pmatrix} \\ &= \begin{pmatrix} G_2^{-1}(\gamma_1) & G_2^{-1}(\gamma_2) & \cdots & G_2^{-1}(\gamma_n) \\ G_2^{-1}(\gamma_1)\gamma_1 & G_2^{-1}(\gamma_2)\gamma_2 & \cdots & G_2^{-1}(\gamma_n)\gamma_n \\ \vdots & \vdots & & \vdots \\ G_2^{-1}(\gamma_1)\gamma_1^{h-2} & G_2^{-1}(\gamma_2)\gamma_2^{h-2} & \cdots & G_2^{-1}(\gamma_n)\gamma_n^{h-2} \\ G_2^{-1}(\gamma_1)\gamma_1^{h-1} & G_2^{-1}(\gamma_2)\gamma_2^{h-1} & \cdots & G_2^{-1}(\gamma_n)\gamma_n^{h-1} \end{pmatrix} \end{aligned}$$

and in particular,

$$P_{2,i} = \begin{pmatrix} G_2^{-1}(\gamma_{i,1}) & \cdots & G_2^{-1}(\gamma_{i,r+\delta-1}) \\ G_2^{-1}(\gamma_{i,1})\gamma_{i,1} & \cdots & G_2^{-1}(\gamma_{i,r+\delta-1})\gamma_{i,r+\delta-1} \\ \vdots & & \vdots \\ G_2-1(\gamma_{i,1})\gamma_{i,1}^{\delta-3} & \cdots & G_2^{-1}(\gamma_{i,r+\delta-1})\gamma_{i,r+\delta-1}^{\delta-3} \\ G_2^{-1}(\gamma_{i,1})\gamma_{i,1}^{\delta-2} & \cdots & G_2^{-1}(\gamma_{i,r+\delta-1})\gamma_{i,r+\delta-1}^{\delta-2} \end{pmatrix}$$

for $1 \leqslant i \leqslant \ell$. Thus, by the fact $G_1(\gamma_{i,j}) \neq 0$ for $1 \leqslant i \leqslant \ell$ and $1 \leqslant j \leqslant r + \delta - 1$, we have the code symbol $v_{i,j}$ has (r, δ)-locality, i.e., the matrix $P_{1,i}$ is a parity-check matrix of a code with minimum Hamming distance at least δ. Now the desired result follows from the fact that the code determined by P has parameters $[n, k \geqslant n - h - \ell(\delta - 1)]_{q^m}$. ■

To bound the Hamming distance of $\Gamma_{q^m}(\mathcal{S}, \mathcal{G})$, we define an auxiliary code over the splitting field of $G_1(x)G_2(x)$. Let $\mathbb{F}_{q^{m_1}}$ be the splitting field of $G_1(x)G_2(x)$ and let $B_1 = \{b_{1,1}, b_{1,2}, \ldots, b_{1,\delta-1}\}$ and $B_2 = \{b_{2,1}, b_{2,2}, \ldots, b_{2,h}\}$ be the roots of $G_1(x)$ and $G_2(x)$ over $\mathbb{F}_{q^{m_1}}$, respectively. Define the code $\Gamma_{q^{m_1}}(\mathcal{S}, \mathcal{G})$ as a set of vectors $V^* = (v_1^*, v_2^*, \ldots, v_n^*) \in \mathbb{F}_{q^{m_1}}^n$ such that

$$\sum_{1\leqslant j\leqslant r+\delta-1} \frac{v_{i,j}^*}{x - \gamma_{i,j}} \equiv 0 \pmod{G_1(x)} \tag{12}$$

and

$$\sum_{1\leqslant j\leqslant n} \frac{v_j^*}{x - \gamma_j} \equiv 0 \pmod{G_2(x)}, \tag{13}$$

where for $1 \leqslant i \leqslant \ell$ and $1 \leqslant j \leqslant r + \delta - 1$, we denote $v_{(i-1)(r+\delta-1)+j}^*$ as $v_{i,j}^*$.

Lemma 10: For the code $\Gamma_{q^{m_1}}(\mathcal{S}, \mathcal{G})$, if $G_1(x)G_2(x)$ has $\delta - 1 + h$ distinct roots over $\mathbb{F}_{q^{m_1}}$, then its parity-check matrix can be written as

$$P^* = \begin{pmatrix} P_{1,1}^* & 0 & \cdots & 0 & 0 \\ 0 & P_{1,2}^* & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & P_{1,\ell}^* & 0 \\ P_{2,1}^* & P_{2,2}^* & \cdots & P_{2,\ell}^* & P_{2,\ell+1}^* \end{pmatrix},$$

where for $1 \leqslant i \leqslant \ell$ $P_{1,i}^* = (p_{t,j}^{(i)})$ is a $(\delta - 1) \times (r + \delta - 1)$ Cauchy matrix with $p_{t,j}^{(i)} = \frac{1}{b_{1,t}-\gamma_{i,j}}$ for $1 \leqslant t \leqslant \delta - 1$ and $1 \leqslant j \leqslant r + \delta - 1$ and $(P_{2,1}^*, P_{2,2}^*, \ldots, P_{2,\ell+1}^*) = (p_{t,j})$ is an $h \times n$ Cauchy matrix with $p_{t,j} = \frac{1}{b_{2,t}-\gamma_j}$ for $1 \leqslant t \leqslant h$ and $1 \leqslant j \leqslant n$. In particular, $P_{2,i}^* = (p_{t,j}^{(i)})$ is a $(\delta-1)\times(r+\delta-1)$ Cauchy matrix with $p_{t,j}^{(i)} = \frac{1}{b_{2,t}-\gamma_{i,j}}$ for $1 \leqslant i \leqslant \ell$, $1 \leqslant t \leqslant \delta - 1$ and $1 \leqslant j \leqslant r + \delta - 1$.

Proof: Obviously, if $V^* \in \Gamma_{q^{m_1}}(\mathcal{S},\mathcal{G})$ is a codeword, then (12) and (13) imply that $P^*V^* = \mathbf{0}$. For any vector $V' \in \mathbb{F}_{q^{m_1}}^n$ with $P^*V' = \mathbf{0}$, we have

$$\sum_{1\leqslant j\leqslant r+\delta-1} \frac{v'_{i,j}}{x-\gamma_{i,j}} \equiv 0 \pmod{x-b_{1,i}}$$

for $1 \leqslant i \leqslant \delta-1$ and

$$\sum_{1\leqslant j\leqslant n} \frac{v'_j}{x-\gamma_j} \equiv 0 \pmod{x-b_{2,j}},$$

for $1 \leqslant j \leqslant h$. Now the fact that $G_1(x)G_2(x)$ has $h+\delta-1$ distinct roots means that

$$\sum_{1\leqslant j\leqslant r+\delta-1} \frac{v'_{i,j}}{x-\gamma_{i,j}} \equiv 0 \pmod{G_1(x) = \prod_{1\leqslant i\leqslant \delta-1}(x-b_{1,i})}$$

and

$$\sum_{1\leqslant j\leqslant n} \frac{v'_j}{x-\gamma_j} \equiv 0 \pmod{G_2(x) = \prod_{1\leqslant i\leqslant h}(x-b_{2,i})},$$

i.e., $V' \in \Gamma_{q^{m_1}}(\mathcal{S},\mathcal{G})$. This completes the proof. ■

Theorem 5: Assume $G_1(x)G_2(x)$ has $\delta-1+h$ distinct roots over $\mathbb{F}_{q^{m_1}}$. For any $t+1$-subset D of $[\ell]$, if

$$\left| S_i \cap \left(\bigcup_{j\neq i, j\in D} S_j \right) \right| \leqslant \delta-1 \quad \text{for } i\in D \tag{14}$$

and

$$S_{\ell+1} \cap S_i = \varnothing \quad \text{for } 1\leqslant i\leqslant \ell, \tag{15}$$

then the code $\Gamma_{q^{m_1}}(\mathcal{S},\mathcal{G})$ has minimum Hamming distance $d \geqslant \min\{(t+1)\delta, h+\delta\}$.

However, before proving the theorem, we first prove two technical lemmas which will be used in the proof.

Lemma 11: Let $\mathcal{E} = \{E_1, E_2, \ldots, E_\tau\}$ be a set of subsets of $\mathbb{F}_{q^{m_1}}$, and let $\Theta = \{\theta_i \;:\; 1\leqslant i\leqslant h_1\} \subseteq \mathbb{F}_{q^{m_1}} \backslash (\bigcup_{1\leqslant j\leqslant \tau} E_j)$. Define an $h_1 \times \tau$ matrix, $M(\mathcal{E},\Theta)$, whose (i,j) entry is

$$M(\mathcal{E},\Theta)_{i,j} = \frac{1}{f_{E_i}(\theta_j)}, \tag{16}$$

where

$$f_{E_i}(x) = \prod_{\theta\in E_i}(x-\theta) \quad \text{for } 1\leqslant i\leqslant \tau.$$

If, for any $t-1$ sets $E_{i_1}, E_{i_2}, \cdots, E_{i_{t-1}} \in \mathcal{E}$, $|\bigcup_{1\leqslant j\leqslant t-1} E_{i_j}| < h_1$, and any $E_i \in \mathcal{E}$ cannot be covered by $t \leqslant \tau-1$ other elements of $\mathcal{E}$, i.e.,

$$E_i \nsubseteq \bigcup_{1\leqslant j\leqslant t, i_j\neq i} E_{i_j} \quad \text{for any } \{i_j \;:\; 1\leqslant j\leqslant t\} \subseteq [\tau]\backslash\{i\}, \tag{17}$$

then any t columns of $M(\mathcal{E},\Theta)$ are linearly independent over $\mathbb{F}_{q^{m_1}}$.

Proof: We assume to the contrary that there exist t columns of $M(\mathcal{E},\Theta)$ that are linearly dependent over $\mathbb{F}_{q^{m_1}}$, which form a sub-matrix of $M(\mathcal{E},\theta)$ given by

$$M' \triangleq \begin{pmatrix} \frac{1}{f_{E_{i_1}}(\theta_1)} & \frac{1}{f_{E_{i_2}}(\theta_1)} & \cdots & \frac{1}{f_{E_{i_t}}(\theta_1)} \\ \frac{1}{f_{E_{i_1}}(\theta_2)} & \frac{1}{f_{E_{i_2}}(\theta_2)} & \cdots & \frac{1}{f_{E_{i_t}}(\theta_2)} \\ \vdots & \vdots & & \vdots \\ \frac{1}{f_{E_{i_1}}(\theta_{h_1})} & \frac{1}{f_{E_{i_2}}(\theta_{h_1})} & \cdots & \frac{1}{f_{E_{i_t}}(\theta_{h_1})} \end{pmatrix},$$

where $\Theta \subseteq \mathbb{F}_{q^{m_1}} \backslash (\bigcup_{1\leqslant j\leqslant \tau} E_j)$ means that M' is well defined. Since $\mathrm{Rank}(M') < t$, there exists a function $f(x) = \sum_{1\leqslant j\leqslant t} e_i \frac{1}{f_{E_{i_j}}(x)}$ where $\theta_1, \ldots, \theta_{h_1}$ are roots of $f(x) = 0$, and where $(e_1, e_2, \ldots, e_t) \neq \mathbf{0}$. Denote $E = \bigcup_{1\leqslant j\leqslant t} E_{i_j}$. It follows that

$$f^*(x) \triangleq f_E(x) \left(\sum_{1\leqslant j\leqslant t} \frac{e_j}{f_{E_{i_j}}(x)} \right) = 0$$

with degree at most $\max\{|\bigcup_{1\leqslant j\neq s_1\leqslant t} E_{i_j}| \;:\; 1\leqslant s_1\leqslant t\} < h_1$ has h_1 roots over $\mathbb{F}_{q^{m_1}}$, which means $f^*(x) = 0$. However, by (17), for any given $1 \leqslant s_1 \leqslant t$ there exists a $\theta \in \Theta$ such that

$$f_{E\backslash E_{i_{s_1}}}(\theta) \neq 0$$

and

$$f_{E\backslash E_{i_{s_2}}}(\theta) = 0 \quad \text{for all } 1\leqslant s_2 \neq s_1 \leqslant t.$$

Thus, $f_{E\backslash E_{i_j}}(\theta)$ for $1 \leqslant j \leqslant t$ are linearly independent over $\mathbb{F}_{q^{m_1}}$, which is a contradiction with $(e_1, e_2, \ldots, e_{t_1}) \neq \mathbf{0}$. ■

Remark 13: When $\delta = 1$, $M(\mathcal{E},\Theta)$ is exactly the well-known Cauchy matrix and the result in Lemma 11 is just the known property of Cauchy matrices.

Lemma 12: Let $W = (\alpha_1, \alpha_2, \cdots, \alpha_n) \in \mathbb{F}_{q^{m_1}}^n$ and let $W^* = \{\alpha_i \;:\; 1\leqslant i\leqslant n\}$. Denote $W_i = \{\alpha_{i,j} \triangleq \alpha_{(i-1)(\tau)+j} \;:\; 1\leqslant j\leqslant \tau\}$ for $1 \leqslant i \leqslant \frac{n}{\tau}$, where τ is an integer factor of n. Let δ be an integer with $\delta \leqslant m_1$, $\Theta_1 = \{\theta_{1,i} \;:\; 1\leqslant i\leqslant \delta-1\} \subseteq \mathbb{F}_{q^{m_1}} \backslash W^*$, $\Theta_2 = \{\theta_{2,i} \;:\; 1\leqslant i\leqslant h\} \subseteq \mathbb{F}_{q^{m_1}} \backslash (W^* \cup \Theta_1)$, and let M be a matrix satisfying

$$M = \begin{pmatrix} M_{1,1} & 0 & \cdots & 0 \\ 0 & M_{1,2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M_{1,\frac{n}{\tau}} \\ M_{2,1} & M_{2,2} & \cdots & M_{2,\frac{n}{\tau}} \end{pmatrix},$$

where for $1 \leqslant i \leqslant \frac{n}{\tau}$ $M_{1,i} = (m_{t,j}^{(i)})$ is a $(\delta-1)\times\tau$ Cauchy matrix with $m_{t,j}^{(i)} = \frac{1}{\theta_{1,t}-\alpha_{i,j}}$ for $1 \leqslant t \leqslant \delta-1$ and $1 \leqslant j \leqslant \tau$ and $(M_{2,1}, M_{2,2}, \ldots, M_{2,\frac{n}{\tau}}) = (m_{t,j})$ is an $h \times n$ Cauchy matrix with $m_{t,j} = \frac{1}{\theta_{2,t}-\alpha_j}$ for $1 \leqslant t \leqslant h$ and $1 \leqslant j \leqslant n$. If $|\Theta_1 \cup \Theta_2| = h+\delta-1$ and $|W_i| = \tau$, then the matrix M can be rewritten as

$$M = LM^*R = \begin{pmatrix} L_1 & 0 & \cdots & 0 & 0 \\ 0 & L_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & L_{\frac{n}{\tau}} & 0 \\ 0 & 0 & \cdots & 0 & I_h \end{pmatrix} \cdot \begin{pmatrix} M_{1,1}^* & 0 & \cdots & 0 \\ 0 & M_{1,2}^* & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M_{1,\frac{n}{\tau}}^* \\ M_{2,1}^* & M_{2,2}^* & \cdots & M_{2,\frac{n}{\tau}}^* \end{pmatrix} \cdot \begin{pmatrix} R_1 & 0 & \cdots & 0 \\ 0 & R_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R_{\frac{n}{\tau}} \end{pmatrix},$$

where, for $1 \leqslant i \leqslant \frac{n}{\tau}$, $|A_i| \neq 0$, $M_{1,i}^* = (I_{\delta-1}, 0_{(\delta-1)\times(\tau-\delta+1)})$ and $M_{2,i}^* = (M_i, M(\mathcal{E}_i, \Theta_2))$ with

$$\mathcal{E}_i = \{E_{i,j} = \{\alpha_{i,1}, \ldots, \alpha_{i,\delta-1}, \alpha_{i,j}\} \;:\; \delta\leqslant j\leqslant \tau\}$$

and $M(\mathcal{E}_i, \Theta_2)$ defined in Lemma 11 by (16).

Proof: We prove this lemma by induction on δ. For the base case we consider the case $\delta = 2$. Note that $\begin{pmatrix} M_{1,i} \\ M_{2,i} \end{pmatrix}$ is a Cauchy matrix for $1 \leqslant i \leqslant \frac{n}{\tau}$. This, together with the facts $|\Theta_1 \cup \Theta_2| = h + \delta - 1$, $|W_i| = \tau$, and $\Theta_1 \cup \Theta_2 \subseteq \mathbb{F}_{q^{m_1}} \backslash W^*$, means that $\begin{pmatrix} M_{1,i} \\ M_{2,i} \end{pmatrix}$ can be rewritten as

$$\begin{pmatrix} M_{1,i} \\ M_{2,i} \end{pmatrix} = \begin{pmatrix} L_i^{(2)} & 0 \\ 0 & I_h \end{pmatrix} \begin{pmatrix} M_{1,i}^{(2)} \\ M_{2,i}^{(2)} \end{pmatrix} R_i^{(1)} \quad \text{for } 1 \leqslant i \leqslant \frac{n}{\tau},$$

where

$$M_{1,i}^{(2)} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \end{pmatrix}_{1 \times \tau}$$

and

$$M_{2,i}^{(2)} = \begin{pmatrix} \frac{1}{\theta_{2,1}-\alpha_{i,1}} & \frac{1}{(\theta_{2,1}-\alpha_{i,1})(\theta_{2,1}-\alpha_{i,2})} & \cdots & \frac{1}{(\theta_{2,1}-\alpha_{i,1})(\theta_{2,1}-\alpha_{i,\tau})} \\ \frac{1}{\theta_{2,2}-\alpha_{i,1}} & \frac{1}{(\theta_{2,2}-\alpha_{i,1})(\theta_{2,2}-\alpha_{i,2})} & \cdots & \frac{1}{(\theta_{2,2}-\alpha_{i,1})(\theta_{2,2}-\alpha_{i,\tau})} \\ \vdots & \vdots & & \vdots \\ \frac{1}{\theta_{2,h}-\alpha_{i,1}} & \frac{1}{(\theta_{2,h}-\alpha_{i,1})(\theta_{2,h}-\alpha_{i,2})} & \cdots & \frac{1}{(\theta_{2,h}-\alpha_{i,1})(\theta_{2,h}-\alpha_{i,\tau})} \end{pmatrix}$$

and the lemma follows for this case.

For the induction hypothesis we assume that the desired results hold for $2 \leqslant \delta \leqslant u$. For the induction step, namely, the case $\delta = u + 1$, similarly, $\Theta_1 \cup \Theta_2 \subseteq \mathbb{F}_{q^{m_1}} \backslash W^*$ means that $\begin{pmatrix} M_{1,i} \\ M_{2,i} \end{pmatrix}$ can be rewritten as

$$\begin{pmatrix} M_{1,i} \\ M_{2,i} \end{pmatrix} = \begin{pmatrix} L_i^{(u+1)} & 0 \\ 0 & I_h \end{pmatrix} \begin{pmatrix} M_{1,i}^{(u+1)} \\ M_{2,i}^{(u+1)} \end{pmatrix} R_i^{(u)} \quad \text{for } 1 \leqslant i \leqslant \frac{n}{\tau}, \tag{18}$$

where

$$M_{1,i}^{(u+1)} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{\theta_{1,2}-\alpha_{i,2}} & \frac{1}{\theta_{1,2}-\alpha_{i,3}} & \cdots & \frac{1}{\theta_{1,2}-\alpha_{i,\tau}} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & \frac{1}{\theta_{1,\delta-1}-\alpha_{i,2}} & \frac{1}{\theta_{1,\delta-1}-\alpha_{i,3}} & \cdots & \frac{1}{\theta_{1,\delta-1}-\alpha_{i,\tau}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & M_{1,i}^{(u)} \end{pmatrix}$$

and

$$M_{2,i}^{(u+1)} = \begin{pmatrix} \frac{1}{\theta_{2,1}-\alpha_{i,1}} & \frac{1}{(\theta_{2,1}-\alpha_{i,1})(\theta_{2,1}-\alpha_{i,2})} & \cdots & \frac{1}{(\theta_{2,1}-\alpha_{i,1})(\theta_{2,1}-\alpha_{i,\tau})} \\ \frac{1}{\theta_{2,2}-\alpha_{i,1}} & \frac{1}{(\theta_{2,2}-\alpha_{i,1})(\theta_{2,2}-\alpha_{i,2})} & \cdots & \frac{1}{(\theta_{2,2}-\alpha_{i,1})(\theta_{2,2}-\alpha_{i,\tau})} \\ \vdots & \vdots & & \vdots \\ \frac{1}{\theta_{2,h}-\alpha_{i,1}} & \frac{1}{(\theta_{2,h}-\alpha_{i,1})(\theta_{2,h}-\alpha_{i,2})} & \cdots & \frac{1}{(\theta_{2,h}-\alpha_{i,1})(\theta_{2,h}-\alpha_{i,\tau})} \end{pmatrix} = T_i^{(u)} \begin{pmatrix} \mathbf{1} & M_{2,i}^{(u)} \end{pmatrix}$$

with $T_i^{(u)} = \mathrm{diag}(\frac{1}{\theta_{2,1}-\alpha_{i,1}}, \frac{1}{\theta_{2,2}-\alpha_{i,1}}, \ldots, \frac{1}{\theta_{2,h}-\alpha_{i,1}})$. By the induction hypothesis,

$$M^{(u)} = \begin{pmatrix} M_{1,1}^{(u)} & 0 & \cdots & 0 \\ 0 & M_{1,2}^{(u)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M_{1,\frac{n}{\tau}}^{(u)} \\ M_{2,1}^{(u)} & M_{2,2}^{(u)} & \cdots & M_{2,\frac{n}{\tau}}^{(u)} \end{pmatrix} = \begin{pmatrix} L_1' & 0 & \cdots & 0 & 0 \\ 0 & L_2' & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & L_{\frac{n}{\tau}}' & 0 \\ 0 & 0 & \cdots & 0 & I_h \end{pmatrix} \cdot \begin{pmatrix} M_{1,1}' & 0 & \cdots & 0 \\ 0 & M_{1,2}' & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M_{1,\frac{n}{\tau}}' \\ M_{2,1}' & M_{2,2}' & \cdots & M_{2,\frac{n}{\tau}}' \end{pmatrix} \cdot \begin{pmatrix} R_1' & 0 & \cdots & 0 \\ 0 & R_2' & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R_{\frac{n}{\tau}}' \end{pmatrix} \tag{19}$$

where for $1 \leqslant i \leqslant \frac{n}{\tau}$, $M_{1,i}' = (I_u, 0_{u \times (\tau-\delta+1)})$ and $M_{2,i}' = (M_i', M(\mathcal{E}_i', \Theta_2))$ with

$$\mathcal{E}_i' = \{E_{i,j}' = \{\alpha_{i,2}, \ldots, \alpha_{i,\delta-1}, \alpha_{i,j}\} \ : \ u+1 \leqslant j \leqslant \tau\}.$$

Combining (18) and (19), we have

$$M = \begin{pmatrix} M_{1,1} & 0 & \cdots & 0 \\ 0 & M_{1,2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M_{1,\frac{n}{\tau}} \\ M_{2,1} & M_{2,2} & \cdots & M_{2,\frac{n}{\tau}} \end{pmatrix} = \begin{pmatrix} L_1 & 0 & \cdots & 0 & 0 \\ 0 & L_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & L_{\frac{n}{\tau}} & 0 \\ 0 & 0 & \cdots & 0 & I_h \end{pmatrix} \cdot \begin{pmatrix} M_{1,1}^* & 0 & \cdots & 0 \\ 0 & M_{1,2}^* & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M_{1,\frac{n}{\tau}}^* \\ M_{2,1}^* & M_{2,2}^* & \cdots & M_{2,\frac{n}{\tau}}^* \end{pmatrix} \cdot \begin{pmatrix} R_1 & 0 & \cdots & 0 \\ 0 & R_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R_{\frac{n}{\tau}} \end{pmatrix}.$$

Here, for $1 \leqslant i \leqslant \frac{n}{\tau}$, $R_i = R_i' R_i^{(u)}$, $L_i = L_i^{(u)} \begin{pmatrix} 1 & 0 \\ 0 & L_i' \end{pmatrix}$, $M_{1,i}^* = \begin{pmatrix} 1 & 0 \\ 0 & M_{1,i}' \end{pmatrix}$ and

$$M_{2,i}^* = T_i^{(u)} \left(\mathbf{1}, M_{2,i}'\right) = \left(T_i^{(u)}\mathbf{1}, T_i^{(u)} M_i', T_i^{(u)} M(\mathcal{E}_i', \Theta_2)\right) = (M_i, M(\mathcal{E}_i, \Theta_2))$$

with $M_i = (T_i^{(u)}\mathbf{1}, T_i^{(u)} M_i')$ and

$$\mathcal{E}_i = \{E_{i,j} = \{\alpha_{i,1}, \ldots, \alpha_{i,\delta-1}, \alpha_{i,j}\} \ : \ u+1 \leqslant j \leqslant \tau\}.$$

By induction, this completes the proof. ■

Proof of Theorem 5: By Lemma 10 the parity-check matrix can be given as

$$P^* = \begin{pmatrix} P^*_{1,1} & 0 & \cdots & 0 & 0 \\ 0 & P^*_{1,2} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & P^*_{1,\ell} & 0 \\ P^*_{2,1} & P^*_{2,2} & \cdots & P^*_{2,\ell} & P^*_{2,\ell+1} \end{pmatrix},$$

where for $1 \leqslant i \leqslant \ell$ $P^*_{1,i} = (p^{(i)}_{u,j})$ is a $(\delta-1) \times (r+\delta-1)$ Cauchy matrix with $p^{(i)}_{u,j} = \frac{1}{b_{1,u}-\gamma_{i,j}}$ for $1 \leqslant u \leqslant \delta-1$ and $1 \leqslant j \leqslant r+\delta-1$ and $(P^*_{2,1}, P^*_{2,2}, \ldots, P^*_{2,\ell+1}) = (p_{u,j})$ is an $h \times n$ Cauchy matrix with $p_{u,j} = \frac{1}{b_{2,u}-\gamma_j}$ for $1 \leqslant u \leqslant h$ and $1 \leqslant j \leqslant n$.

We consider the case that there are at most $e \leqslant \min\{(t+1)\delta-1, h+\delta-1\}$ erasures in total, i.e., $e = \sum_{1\leqslant i\leqslant \ell+1} |E_i| \leqslant \min\{t\delta, h+\delta-1\}$. To bound the Hamming distance we only need to consider erasure patterns such that $E_i \subseteq S_i$ for $1 \leqslant i \leqslant \ell+1$ and $|E_i| \geqslant \delta$ for $1 \leqslant i \leqslant \ell$. Let $P^*(\mathcal{E})$ be the sub-matrix formed by the columns corresponding to E_i $1 \leqslant i \leqslant \ell+1$, that is the column $(0,\ldots,0, \frac{1}{b_{1,1}-\gamma_{i,j}}, \ldots, \frac{1}{b_{1,\delta-1}-\gamma_{i,j}}, 0\ldots,0, \frac{1}{b_{2,1}-\gamma_{i,j}}, \ldots, \frac{1}{b_{2,h}-\gamma_{i,j}})^{\mathsf{T}}$ is chosen if $\gamma_{i,j} \in E_i \subseteq S_i$. It is easy to check that $P^*(\mathcal{E})$ can be written as

$$P^*(\mathcal{E}) = \begin{pmatrix} P^{\mathcal{E}}_{1,i_1} & 0 & \cdots & 0 & 0 \\ 0 & P^{\mathcal{E}}_{1,i_2} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & P^{\mathcal{E}}_{1,i_{t_1}} & 0 \\ P^{\mathcal{E}}_{2,i_1} & P^{\mathcal{E}}_{2,i_2} & \cdots & P^{\mathcal{E}}_{2,i_{t_1}} & P^{\mathcal{E}}_{2,\ell+1} \end{pmatrix},$$

by deleting the all zero rows. For the case $t_1 = 0$, $\mathrm{Rank}(P^*(\mathcal{E})) = \mathrm{Rank}(P^{\mathcal{E}}_{2,\ell+1}) = |E_{\ell+1}|$ the erasure pattern can be recovered. For the case $t_1 \geqslant 1$, the fact that $(P^*_{2,1}, P^*_{2,2}, \ldots, P^*_{2,\ell+1}) = (p_{u,j})$ is an $h \times n$ Cauchy matrix with $p_{u,j} = \frac{1}{b_{2,u}-\gamma_j}$ for $1 \leqslant u \leqslant h$ and $1 \leqslant j \leqslant n$ means that

$$\begin{aligned} &\mathrm{Rank}(P^*(\mathcal{E})) \\ =&\mathrm{Rank}\begin{pmatrix} P^{\mathcal{E}}_{1,i_1} & 0 & \cdots & 0 & 0 \\ 0 & P^{\mathcal{E}}_{1,i_2} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & P^{\mathcal{E}}_{1,i_{t_1}} & 0 \\ P^{\mathcal{E}}_{2,i_1} & P^{\mathcal{E}}_{2,i_2} & \cdots & P^{\mathcal{E}}_{2,i_{t_1}} & P^{\mathcal{E}}_{2,\ell+1} \end{pmatrix} \\ \geqslant&\mathrm{Rank}\begin{pmatrix} P^{\mathcal{E}}_{1,i_1} & 0 & \cdots & 0 & 0 \\ 0 & P^{\mathcal{E}}_{1,i_2} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & P^{\mathcal{E}}_{1,i_{t_1}} & 0 \\ P^{\mathcal{E},h_1}_{2,i_1} & P^{\mathcal{E},h_1}_{2,i_2} & \cdots & P^{\mathcal{E},h_1}_{2,i_{t_1}} & 0 \\ 0 & 0 & \cdots & 0 & I_{|E_{\ell+1}|} \end{pmatrix}, \end{aligned}$$

where $h_1 = h - |E_{\ell+1}|$ and $P^{\mathcal{E},h_1}_{2,i_j}$ is the sub-matrix formed by the first h_1 rows of $P^{\mathcal{E}}_{2,i_j}$ for $1 \leqslant j \leqslant t_1$.

Recall that $e = \sum_{E\in\mathcal{E}} |E| \leqslant \min\{(t+1)\delta-1, h+\delta-1\}$, which means

$$t_1 \leqslant \begin{cases} \left\lfloor \frac{(t+1)\delta-1-|E_{\ell+1}|}{\delta} \right\rfloor \leqslant t, \\ \qquad \text{if } h_1+\delta-1 \geqslant (t+1)\delta-1-|E_{\ell+1}|, \\ \left\lfloor \frac{h_1+\delta-1}{\delta} \right\rfloor < t, \\ \qquad \text{if } h_1+\delta-1 < (t+1)\delta-1-|E_{\ell+1}|. \end{cases} \tag{20}$$

According to (14), for $i \in \{i_j \ : \ 1 \leqslant j \leqslant t_1\}$

$$\left| E_i \cap \left(\bigcup_{\substack{1\leqslant j\leqslant t_1 \\ i_j \neq i}} E_{i_j} \right) \right| \leqslant \left| S_i \cap \left(\bigcup_{\substack{1\leqslant j\leqslant t_1 \\ i_j \neq i}} S_{i_j} \right) \right| \leqslant \delta-1,$$

which means that the elements of each E_i may be indexed $E_i = \{\alpha_{i,u} \ : \ 1 \leqslant u \leqslant \tau_i\}$ such that

$$\{\alpha_{i,t} \ : \ \delta \leqslant t \leqslant \tau_i\} \cap E_{i_j} = \varnothing \text{ for } 1 \leqslant j \leqslant t_1, \ i_j \neq i.$$

For $1 \leqslant j \leqslant t_1$, let $\mathcal{E}_{i_j} = \{\{\alpha_{i_j,1}, \ldots, \alpha_{i_j,\delta-1}, \alpha_{i_j,u}\} \ : \ \delta \leqslant u \leqslant \tau_{i_j}\}$ and $\mathcal{E}^* = \bigcup_{1\leqslant j\leqslant t_1} \mathcal{E}_{i_j}$. By Lemma 12,

$$\mathrm{Rank}(P^*(\mathcal{E})) \geqslant t_1(\delta-1) + |E_{\ell+1}| + \mathrm{Rank}(M(\mathcal{E}^*, \Theta_3)),$$

where $\Theta_3 = \{\gamma_{2,i} \ : \ 1 \leqslant i \leqslant h_1\}$. Thus, by Lemma 11, (14), (15), and (20), $M(\mathcal{E}^*, \Theta_3)$ has full rank, i.e., $\mathrm{Rank}(P^*(\mathcal{E})) \geqslant |E_{\ell+1}| + \sum_{1\leqslant j\leqslant t_1} |E_{i_j}|$. This is to say, the erasure pattern can be recovered, which means $d \geqslant \min\{(t+1)\delta, h+\delta\}$. ■

Corollary 12: Assume $G_1(x)G_2(x)$ has $\delta-1+h$ distinct roots over $\mathbb{F}_{q^{m_1}}$. Let $\mathcal{S}$ be a set system of $\mathbb{F}_{q^m}$ such that for any $t+1$-subset D of $[\ell]$

$$\left| S_i \cap \left(\bigcup_{j\neq i, j\in D} S_j \right) \right| \leqslant \delta-1 \quad \text{for } i \in D$$

and

$$S_{\ell+1} \cap S_i = \varnothing \quad \text{for } 1 \leqslant i \leqslant \ell.$$

If $h+\delta \leqslant (t+1)\delta$ and $S_{\ell+1} \neq \varnothing$, then the code $\Gamma_{q^m}(\mathcal{S}, \mathcal{G})$ is an optimal $[n,k,d=h+\delta]_{q^m}$ linear code with $(r,\delta)_i$-locality.

Proof: By Theorem 5, the facts $\Gamma_{q^m}(\mathcal{S}, \mathcal{G}) \subseteq \Gamma_{q^{m_1}}(\mathcal{S}, \mathcal{G})$ and $h+\delta \leqslant (t+1)\delta$ show that $\Gamma_{q^m}(\mathcal{S}, \mathcal{G})$ has minimum Hamming distance at least $h+\delta$. Thus, by Lemma 9, $\Gamma_{q^m}(\mathcal{S}, \mathcal{G})$ is an $[n,k,d \geqslant h+\delta]_{q^m}$ with $k \geqslant n-\ell(\delta-1)-h$ and those symbols with (r,δ)-locality have rank at least $k_1 = n-\ell(\delta-1)-h = \ell r$. By Lemma 1,

$$\begin{aligned} d \leqslant& n-k+1-\left(\left\lceil \frac{k_1}{r} \right\rceil - 1\right)(\delta-1) \\ \leqslant& \ n-k+1-(\ell-1)(\delta-1) \leqslant h+\delta, \end{aligned}$$

which together with the fact $d \geqslant h+\delta$ show that $d = h+\delta$ and $k = k_1$. This is also to say that $\Gamma_{q^m}(\mathcal{S}, \mathcal{G})$ is an optimal linear code with $(r,\delta)_i$-locality with respect to the bound in Lemma 1. ■

For the case $S_{\ell+1} = \varnothing$, the following corollary follows directly from Theorem 5 and Lemma 9.

Corollary 13: Let $n = \ell(r+\delta-1)$ and $0 < h \leqslant r$. Assume $G_1(x)G_2(x)$ has $\delta-1+h$ distinct roots over $\mathbb{F}_{q^{m_1}}$. Let $\mathcal{S}$ be a set system of $\mathbb{F}_{q^m}$ such that for any $t+1$-subset D of $[\ell]$

$$\left| S_i \cap \left(\bigcup_{j\neq i, j\in D} S_j \right) \right| \leqslant \delta - 1 \quad \text{for } i \in D$$

If $h+\delta \leqslant (t+1)\delta$ and $S_{\ell+1} = \varnothing$, then the code $\Gamma_{q^m}(\mathcal{S}, \mathcal{G})$ is an $[n = \ell(r+\delta-1), k, d \geqslant h+\delta]_{q^m}$ code with $(r,\delta)_a$-locality, where

$$n - \ell(\delta-1) = \ell r \geqslant k \geqslant \ell r - h.$$

Furthermore, if $k = \ell r - h$, then $\Gamma_{q^m}(\mathcal{S}, \mathcal{G})$ is an $[n, k, d = h+\delta]$ optimal locally repairable code with $(r,\delta)_a$-locality.

Remark 14: For the case $k = \ell r - h$ and $h \leqslant r$, i.e., $S_{\ell+1} = \emptyset$, the codes generated by Construction E share similar parameters with those constructed in [11]. However, Construction E may also work for the case of $S_{\ell+1} \neq \emptyset$ in which we may construct optimal locally repairable codes with new parameters as shown in Corollary 12.

VI. Conclusion

In this article, we first introduced a construction of locally repairable codes with $(r,\delta)_i$-locality. To analyze the performance of our construction, an upper bound was derived for the length of optimal locally repairable codes with $(r,\delta)_i$-locality. Our main goal, with this bound, is to find a connection between the length of the code and the field size over which the code is constructed. Using combinatorial structures (packings in general, and Steiner systems in particular) we arrive at the conclusion that, in some cases, the optimal locally-repairable codes we constructed have order-optimal length, which is super-linear in the field size. We also suggested another construction for optimal locally repairable codes, this time, taking inspiration from Goppa codes. The construction share a similarity in the combinatorial structures they require. Finally, we defined generalized sector-disk codes. We showed that the locally repairable codes of our constructions are capable of yielding GSD codes, and compared their parameters with sector-disk (SD) codes, and maximally recoverable (MR) codes.

In general, it seems that constructions of locally repairable codes have focused mainly on $(r,\delta)_a$-locality, perhaps due to their symmetry. We believe our constructions and bound show that codes with $(r,\delta)_i$-locality are also of theoretical and applicative interest. Several open questions remain, including finding SD/MR/GSD codes for all possible parameters, and finding more codes that are capable of correcting special erasure patterns beyond what is guaranteed due to their Hamming distance.

Appendix A
Proof of Theorem 3

Lemma 13: Let $\mathcal{C}$ be an $[n,k]_q$ linear code with $(r,\delta)_i$-locality and $r|k$. Let $\mathcal{A}$ be the set of all the repair sets of information symbols, where we highlight that there may exist some information symbols that share the same repair set. For any $1 \leqslant j \leqslant \frac{k}{r}$, if there exists a j-subset $\mathcal{V} \subseteq \mathcal{A}$ and $\Delta > 0$ is an integer such that for any $A \in \mathcal{V}$

$$\left| A \cap \left(\bigcup_{A' \in \mathcal{V}\setminus\{A\}} A' \right) \right| \leqslant |A| - \delta + 1 \tag{21}$$

and

$$|\mathcal{V}|(r+\delta-1) - \left| \bigcup_{A\in\mathcal{V}} A \right| \geqslant \Delta > 0,$$

then there exists a set $S \subseteq [n]$ with $\mathrm{Rank}(S) = k-1$ and

$$|S| \geqslant k - 1 + \frac{k}{r}(\delta - 1).$$

Proof: Let $\mathcal{V} = \{A_{i_1}, A_{i_2}, \ldots, A_{i_j}\}$ and

$$A^*_{i_t} \subseteq A_{i_t} \setminus \left(\bigcup_{A' \in \mathcal{V}\setminus\{A_{i_t}\}} A' \right) \tag{22}$$

with $|A^*_{i_t}| = \delta - 1$ for $1 \leqslant t \leqslant j$, which is possible in light of (21). Define a $(|A_{i_t}| - \delta + 1)$-subset $A'_{i_t} \triangleq A_{i_t} \setminus A^*_{i_t}$ for $1 \leqslant t \leqslant j$. By definition 1, we have $\mathrm{Rank}(A'_{i_t}) = \mathrm{Rank}(A_{i_t})$ for $1 \leqslant t \leqslant j$. Note that (22) implies that $A^*_{i_t}$ for $1 \leqslant t \leqslant j$ are pairwise disjoint, which also means that

$$\begin{aligned}
\mathrm{Rank}\left(\bigcup_{1\leqslant t\leqslant j} A_{i_t} \right) &= \mathrm{Rank}\left(\bigcup_{1\leqslant t\leqslant j} (A_{i_t} \setminus A^*_{i_t}) \right) \\
&\leqslant \left| \bigcup_{1\leqslant t\leqslant j} A_{i_t} \setminus A^*_{i_t} \right| \\
&= \left| \bigcup_{1\leqslant t\leqslant j} A_{i_t} \right| - \left| \bigcup_{1\leqslant t\leqslant j} A^*_{i_t} \right| \\
&= \left| \bigcup_{1\leqslant t\leqslant j} A_{i_t} \right| - j(\delta-1),
\end{aligned} \tag{23}$$

where the second equality holds by (22). This is to say that

$$jr - \mathrm{Rank}\left(\bigcup_{1\leqslant t\leqslant j} A_{i_t} \right) \geqslant j(r+\delta-1) - \left| \bigcup_{1\leqslant t\leqslant j} A_{i_t} \right| = \Delta > 0,$$

i.e., $\mathrm{Rank}\left(\bigcup_{1\leqslant t\leqslant j} A_{i_t} \right) \leqslant jr - 1$.

For the case $j < \frac{k}{r}$, the fact that $\mathcal{C}$ has $(r,\delta)_i$-locality, i.e., $\mathrm{Rank}\left(\bigcup_{A\in\mathcal{A}} A \right) = k$ means that there exists an $A_{i_{j+1}}$ such that $\mathrm{Rank}\left(\bigcup_{1\leqslant t\leqslant j+1} A_{i_t} \right) > \mathrm{Rank}\left(\bigcup_{1\leqslant t\leqslant j+1} A_{i_t} \right)$. This is to say $\left| A_{j+1} \cap \left(\bigcup_{1\leqslant t\leqslant j+1} A_{i_t} \right) \right| \leqslant |A_{j+1}| - \delta + 1$. Let $A^*_{i_{j+1}} \subseteq A_{i_{j+1}} \setminus \left(\bigcup_{1\leqslant t\leqslant j+1} A_{i_t} \right)$ with $|A^*_{i_{j+1}}| = \delta - 1$ and $A'_{i_{j+1}} = A_{i_{j+1}} \setminus A^*_{i_{j+1}}$. Note that $A_{i_{j+1}}$ is a repair set of $\mathcal{C}$. Thus, $\mathrm{Rank}(A'_{i_{j+1}}) = \mathrm{Rank}(A_{i_{j+1}})$ by Definition 1 and

$$\begin{aligned}
&\mathrm{Rank}\left(\bigcup_{1\leqslant t\leqslant j+1} A_{i_t} \right) \\
&= \mathrm{Rank}\left(A'_{i_{j+1}} \cup \left(\bigcup_{1\leqslant t\leqslant j} A_{i_t} \right) \right)
\end{aligned}$$

$$\begin{aligned}
&\leqslant \mathrm{Rank}\left(\bigcup_{1\leqslant t\leqslant j} A_{i_t}\right) + \left|A'_{i_{j+1}} \setminus \left(\bigcup_{1\leqslant t\leqslant j} A_{i_t}\right)\right| \\
&\leqslant \mathrm{Rank}\left(\bigcup_{1\leqslant t\leqslant j} A_{i_t}\right) + \left|A_{i_{j+1}} \setminus \left(\bigcup_{1\leqslant t\leqslant j} A_{i_t}\right)\right| - \delta + 1 \\
&= \left|\bigcup_{1\leqslant t\leqslant j+1} A_{i_t}\right| - (j+1)(\delta-1),
\end{aligned}$$

where the last equality holds by (23). Recall that $\mathrm{Rank}(\bigcup_{1\leqslant t\leqslant j} A_{i_t}) < jr$ which means that $\mathrm{Rank}(\bigcup_{1\leqslant t\leqslant j+1} A_{i_t}) < (j+1)r$.

Repeat the preceding analysis $\frac{k}{r} - j$ times, then we can find A_{i_t} with $1 \leqslant t \leqslant \frac{k}{r}$ such that

$$\left|\bigcup_{1\leqslant t \frac{k}{r}} A_{i_t}\right| - \mathrm{Rank}\left(\bigcup_{1\leqslant t \frac{k}{r}} A_{i_t}\right) \geqslant \frac{k}{r}(\delta - 1)$$

and $\mathrm{Rank}\left(\bigcup_{1\leqslant t \frac{k}{r}} A_{i_t}\right) < k$. Thus, we can extend the set $\bigcup_{1\leqslant t\leqslant \frac{k}{r}} A_{i_t}$ to be a set S with $\mathrm{Rank}(S) = k-1$ and

$$\begin{aligned}
|S| - \mathrm{Rank}(S) &= |S| - k + 1 \\
&\geqslant \left|\bigcup_{1\leqslant t\leqslant \frac{k}{r}} A_{i_t}\right| - \mathrm{Rank}\left(\bigcup_{1\leqslant t\leqslant \frac{k}{r}} A_{i_t}\right) \geqslant \frac{k}{r}(\delta - 1),
\end{aligned}$$

which means the desired result follows. ■

Theorem 6: Let $\mathcal{C}$ be an optimal $[n, k, d]_q$ linear code with $(r, \delta)_i$-locality. If $r|k$ and $r < k$, then there exist $\frac{k}{r}$ repair sets $\mathcal{V} = \{A_{i_1}, A_{i_2}, \ldots, A_{i_{\frac{k}{r}}}\}$, such that $|A_{i_t}| = r + \delta - 1$, A_{i_t} for $1 \leqslant t \leqslant \frac{k}{r}$ are pairwise disjoint and $\mathrm{Rank}(\bigcup_{1\leqslant t\leqslant \frac{k}{r}} A_{i_t}) = k$. Furthermore, the punctured code $\mathcal{C}|_{A_{i_t}}$ for $1 \leqslant t \leqslant \frac{k}{r}$ is an $[r + \delta - 1, r, \delta]_q$ MDS code.

Proof: Since the code $\mathcal{C}$ has $(r, \delta)_i$-locality, we have $\mathrm{Rank}(\bigcup_{A\in\mathcal{A}} A) = k$, where $\mathcal{A}$ denotes the set of all repair sets of information symbols. Note that for each repair set $A \in \mathcal{A}$, by Definition 1, we have $\mathrm{Rank}(A) \leqslant r$. This means that we can find A_{i_t} for $1 \leqslant t \leqslant \frac{k}{r}$ such that $\mathrm{Rank}\left(\bigcup_{1\leqslant t\leqslant j} A_{i_t}\right) > \mathrm{Rank}\left(\bigcup_{1\leqslant t\leqslant j-1} A_{i_t}\right)$ for $2 \leqslant j \leqslant \frac{k}{r}$. We claim that those $\frac{k}{r}$ repair sets are pairwise disjoint and $|A_{i_t}| = r + \delta - 1$ for $1 \leqslant t \leqslant \frac{k}{r}$. Note that for $j > t$ we have $|A_{i_t} \cap A_{i_j}| \leqslant |A_j| - \delta + 1$, since $\mathrm{Rank}\left(\bigcup_{1\leqslant t\leqslant j} A_{i_t}\right) > \mathrm{Rank}\left(\bigcup_{1\leqslant t\leqslant j-1} A_{i_t}\right)$. Now by Lemma 13, if $2(r + \delta - 1) - |A_{i_t} \cup A_{i_j}| > 0$ then we have a set S with rank $k - 1$ and $|S| = k - 1 + \frac{k}{r}(\delta - 1)$, which contradicts with the fact that $\mathcal{C}$ is optimal, i.e., $d = n - k + 1 - (\frac{k}{r} - 1)(\delta - 1)$. Thus, for $j > t$ and $1 \leqslant j, t \leqslant \frac{k}{r}$, we have $2(r + \delta - 1) - |A_{i_t} \cup A_{i_j}| = 0$, i.e., $A_{i_t} \cap A_{i_j} = \varnothing$ and $|A_{i_t}| = |A_{i_j}| = r + \delta - 1$, since $|A_{i_t}| \leqslant r + \delta - 1$ and $|A_{i_j}| \leqslant r + \delta - 1$.

Now, we only need to prove that $\mathrm{Rank}\left(\bigcup_{1\leqslant t\leqslant \frac{k}{r}} A_{i_t}\right) = k$. If that is not the case, then we have $\mathrm{Rank}\left(\bigcup_{1\leqslant t\leqslant \frac{k}{r}} A_{i_t}\right) \leqslant k - 1$. Note that $\left|\bigcup_{1\leqslant t\leqslant \frac{k}{r}} A_{i_t}\right| = k + \frac{k}{r}(\delta - 1)$, which is also a contradiction with $d = n - k + 1 - (\frac{k}{r} - 1)(\delta - 1)$. Therefore, the desired result follows. Finally, for $1 \leqslant t \leqslant \frac{k}{r}$, the fact that $\mathrm{Rank}(A_{i_t}) = r$, $|A_{i_t}| = r + \delta - 1$, and $d(\mathcal{C}|_{A_{i_t}}) \geqslant \delta$, shows that $\mathcal{C}|_{A_{i_t}}$ is an $[r + \delta - 1, r, \delta]_q$ MDS code. ■

We are now in a position to prove Theorem 3.

Proof: By Theorem 6, and up to a rearrangement of the code coordinates, the parity-check matrix P of code $\mathcal{C}$ can be arranged in the following form,

$$P = \begin{pmatrix}
L^{(1)} & 0 & 0 & \ldots & 0 & 0 \\
0 & L^{(2)} & 0 & \ldots & 0 & 0 \\
0 & 0 & L^{(3)} & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & L^{(\ell)} & 0 \\
H_1 & H_2 & H_3 & \ldots & H_\ell & H_{\ell+1}
\end{pmatrix},$$

where $L^{(i)} = (I_{\delta-1}, P_i)$ is a $(\delta - 1) \times (r + \delta - 1)$ matrix for all $1 \leqslant i \leqslant w$ and we do row linear transformations to make sure each $L^{(i)}$ has canonical form. Define

$$M_1 \triangleq \begin{pmatrix}
I_{\delta-1} & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & I_{\delta-1} & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & I_{\delta-1} & 0 & 0 \\
0 & H_1^{(1)} & 0 & H_2^{(1)} & \ldots & 0 & H_\ell^{(1)} & H_{\ell+1}
\end{pmatrix} \quad (24)$$

where $I_{\delta-1}$ denotes the $(\delta - 1) \times (\delta - 1)$ identity matrix and $H_i^{(1)} = H_{i,2} - H_{i,1}P_i$ with $H_i = (H_{i,1}, H_{i,2})$ and $1 \leqslant i \leqslant \ell$. For any integer $0 \leqslant a \leqslant h$, let

$$M_{2,a} = \begin{pmatrix} H_1^{(1)} & H_2^{(1)} & H_3^{(1)} & \ldots & H_\ell^{(1)} & H_{\ell+1}^{(a)} \end{pmatrix},$$

where $H_{\ell+1}^{(a)}$ denotes the matrix generated by deleting any a columns from $H_{\ell+1}$.

Now, for any $0 \leqslant a \leqslant h$, the fact that any $d - 1$ columns of P are linearly independent over $\mathbb{F}_q$ means that any $T(a) = \lfloor \frac{d-a-1}{\delta} \rfloor$ columns of M_2 are linearly independent over $\mathbb{F}_q$. This is because any $T(a)$ columns of $M_{2,a}$ correspond to at most $T(a)\delta$ columns of P by adding the first $\delta - 1$ columns in related blocks, and by (24) they have full column rank. Therefore, $M_{2,a}$ is the parity-check matrix of a linear code $\mathcal{C}_{1,a}$, with parameters $[\ell r + h - a, k' \geqslant k = \ell r, d_2 \geqslant T(a) + 1]_q$.

In what follows, we distinguish between two cases, depending on the parity of $T(a)$.

Case 1: $T(a)$ is odd. In this case, we consider the shortened code $\mathcal{C}_{2,a}$ of $\mathcal{C}_{1,a}$ with parameters $[\ell r + h - a - 1, k' \geqslant \ell r, d_2 \geqslant t]_q$. By the Hamming bound [32] we have

$$\begin{aligned}
q^{\ell r} &\leqslant \frac{q^{\ell r + h - a - 1}}{\sum_{0\leqslant i\leqslant \frac{T(a)-1}{2}} \binom{\ell r + h - a - 1}{i} (q-1)^i} \\
&\leqslant \frac{q^{\ell r + h - a - 1}}{\binom{\ell r + h - 1}{\frac{T(a)-1}{2}} (q-1)^{\frac{T(a)-1}{2}}} \\
&\leqslant \frac{q^{\ell r + h - a - 1}}{\left(\frac{\ell r + h - a - 1}{\frac{T(a)-1}{2}}\right)^{\frac{T(a)-1}{2}} (q-1)^{\frac{T(a)-1}{2}}},
\end{aligned}$$

which means

$$\ell r + h - a - 1 \leqslant \frac{T(a) - 1}{2(q-1)} q^{\frac{2(h-a-1)}{T(a)-1}}.$$

This is to say,

$$\begin{aligned} n \leqslant & \frac{r+\delta-1}{r}\left(\frac{T(a)-1}{2(q-1)}q^{\frac{2(h-a-1)}{T(a)-1}}-h+a+1\right)+h \\ = & \frac{r+\delta-1}{r}\left(\frac{T(a)-1}{2(q-1)}q^{\frac{2(h-a-1)}{T(a)-1}}+a+1\right)-\frac{h(\delta-1)}{r}. \end{aligned}$$

Case 2: $T(a)$ is even. Similarly, by the Hamming bound, we have

$$\begin{aligned} q^{\ell r} \leqslant & \frac{q^{\ell r+h-a}}{\sum_{1\leqslant i\leqslant \frac{T(a)}{2}}\binom{\ell r+h-a}{i}(q-1)^i} \leqslant \frac{q^{\ell r+h-a}}{\binom{\ell r+h-a}{\frac{T(a)}{2}}(q-1)^{\frac{T(a)}{2}}} \\ \leqslant & \frac{q^{\ell r+h-a}}{\left(\frac{\ell r+h-a}{\frac{T(a)}{2}}\right)^{\frac{T(a)}{2}}(q-1)^{\frac{T(a)}{2}}}, \end{aligned}$$

which means

$$\begin{aligned} n = & \ell(r+\delta-1)+h \\ \leqslant & \frac{r+\delta-1}{r}\left(\frac{T(a)}{2(q-1)}q^{\frac{2(h-a)}{T(a)}}+a\right)-\frac{h(\delta-1)}{r}. \end{aligned}$$

Finally, recall that by Lemma 1, $\mathcal{C}$ is optimal means that $h=d-\delta$. This completes the proof. ■

Appendix B
Regular Packings

We present a direct construction of regular packings based on a kind of cyclotomy. The generated regular packings are not new, and may obtained recursively via [26], and via generalized cyclotomy [16], [49]. Thus, the construction and proof herein are brought for the reader's convenience only.

According to the unique factorization theorem, a positive integer n has the following unique decomposition

$$n = p_1^{m_1}p_2^{m_2}\cdots p_u^{m_u},$$

where $p_1<p_2<\cdots<p_u$ are primes and $m_1,m_2,\dots,m_u$ are positive integers. For $1\leqslant i\leqslant u$, let $\mathbb{F}_{p_i^{m_i}}$ be the finite field with size $p_i^{m_i}$ and α_i be one of its primitive elements. Let e be a positive integer with

$$e \mid \gcd(p_1^{m_1}-1, p_2^{m_2}-1,\cdots,p_u^{m_u}-1).$$

For $e>1$, define

$$\boldsymbol{\beta}_e \triangleq (\alpha_1^{\frac{p^{m_1}-1}{e}}, \alpha_2^{\frac{p^{m_2}-2}{e}},\dots,\alpha_u^{\frac{p^{m_u}-1}{e}}) \in T,$$

where

$$T \triangleq \mathbb{F}_{p_1^{m_1}}\times\mathbb{F}_{p_2^{m_2}}\times\cdots\times\mathbb{F}_{p_u^{m_u}}.$$

It is easy to verify that $D_0=\langle\boldsymbol{\beta}_e\rangle=\{\boldsymbol{\beta}_e^0,\boldsymbol{\beta}_e^1,\cdots,\boldsymbol{\beta}_e^{e-1}\}\subseteq T^*\triangleq\mathbb{F}^*_{p_1^{m_1}}\times\mathbb{F}^*_{p_2^{m_2}}\times\cdots\times\mathbb{F}^*_{p_u^{m_u}}$ is a subgroup of $(T^*,\cdot)$ with order e, where

$$\boldsymbol{\beta}_e^i \triangleq (\alpha_1^{i\frac{p^{m_1}-1}{e}}, \alpha_2^{i\frac{p^{m_2}-2}{e}},\dots,\alpha_u^{i\frac{p^{m_u}-1}{e}}) \in T.$$

For $J\in A\triangleq\mathbb{Z}_{\frac{p^{m_1}-1}{e}}\times\mathbb{Z}_{\frac{p^{m_1}-1}{e}}\times\dots\mathbb{Z}_{\frac{p^{m_u}-1}{e}}$, define $B_J\subseteq\mathbb{Z}_e\times T$ as

$$B_J \triangleq \{(0,\boldsymbol{\alpha}^J\boldsymbol{\beta}_e^0),(1,\boldsymbol{\alpha}^J\boldsymbol{\beta}_e^1),\dots,(e-1,\boldsymbol{\alpha}^J\boldsymbol{\beta}_e^{e-1})\}, \quad (25)$$

where $\boldsymbol{\alpha}\triangleq(\alpha_1,\alpha_2,\dots,\alpha_u)$ and $\boldsymbol{\alpha}^J\triangleq(\alpha_1^{j_1},\alpha_2^{j_2},\dots,\alpha_u^{j_u})$ for $J=(j_1,j_2,\dots,j_u)$. Based on B_Js, we can generate a set system as:

Construction F: Let $X=\mathbb{Z}_e\times T$, then we may construct a set

$$\mathcal{B}=\{B_{J,\epsilon}=B_J+(0,\epsilon) \ : \ J\in A,\epsilon\in T\}. \quad (26)$$

Theorem 7: The set system $(X,\mathcal{B})$ generated by Construction F is a $\frac{\prod_{1\leqslant i\leqslant u}(p_1^{m_i}-1)}{e^u}$-regular packing with parameters $(en,e,1)$.

Proof: By Construction F, it is sufficient to prove that any pair of elements of X appears in at most one of the blocks in $\mathcal{B}$. Assume to the contrary that there exists a pair $\{x_1=(i_1,\gamma_1),x_2=(i_2,\gamma_2)\}\subseteq X$ that appears in two blocks, i.e., $\{x_1,x_2\}\subseteq B_{J_1,\epsilon_1}$ and $\{x_1,x_2\}\subseteq B_{J_2,\epsilon_2}$, where $i_1,i_2\in\mathbb{Z}_e$ and $\gamma_1,\gamma_2\in T$. By (25) and (26), there exist four elements $t_{1,1},t_{1,2},t_{2,1},t_{2,2}\in\mathbb{Z}_e$ such that

$$\begin{aligned} (0,\epsilon_1)+(t_{1,1},\boldsymbol{\alpha}^{J_1}\boldsymbol{\beta}_e^{t_{1,1}}) = & (i_1,\gamma_1) \\ = & (0,\epsilon_2)+(t_{2,1},\boldsymbol{\alpha}^{J_2}\boldsymbol{\beta}_e^{t_{2,1}}) \quad (27) \end{aligned}$$

and

$$\begin{aligned} (0,\epsilon_1)+(t_{1,2},\boldsymbol{\alpha}^{J_1}\boldsymbol{\beta}_e^{t_{1,2}}) = & (i_2,\gamma_2) \\ = & (0,\epsilon_2)+(t_{2,2},\boldsymbol{\alpha}^{J_2}\boldsymbol{\beta}_e^{t_{2,2}}). \quad (28) \end{aligned}$$

These equalities imply that $t_{1,1}=t_{2,1}$, $t_{1,2}=t_{2,2}$ and

$$\boldsymbol{\alpha}^{J_1}\boldsymbol{\beta}_e^{t_{1,2}-t_{1,1}}=\boldsymbol{\alpha}^{J_2}\boldsymbol{\beta}_e^{t_{2,2}-t_{2,1}},$$

i.e.,

$$\boldsymbol{\alpha}^{J_1}=\boldsymbol{\alpha}^{J_2}. \quad (29)$$

Note that $J_1,J_2\in A=\mathbb{Z}_{\frac{p^{m_1-1}}{e}}\times\mathbb{Z}_{\frac{p^{m_1-1}}{e}}\times\dots\mathbb{Z}_{\frac{p^{m_u-u}}{e}}$ and $\boldsymbol{\alpha}=(\alpha_1,\alpha_2,\cdots,\alpha_u)$, where α_i is a primitive element of $\mathbb{F}_{p_i^{m_i}}$. Thus, by (29), we have $J_1=J_2$. Again by (27) and (28), we have $\epsilon_1=\epsilon_2$, a contradiction. Thus, the desired result follows. ■

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