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On tilings of asymmetric limited-magnitude balls ☆

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ABSTRACT

We study whether an asymmetric limited-magnitude ball may tile $\mathbb{Z}^n$. This ball generalizes previously studied shapes: crosses, semi-crosses, and quasi-crosses. Such tilings act as perfect error-correcting codes in a channel which changes a transmitted integer vector in a bounded number of entries by limited-magnitude errors.

A construction of lattice tilings based on perfect codes in the Hamming metric is given. Several non-existence results are proved, both for general tilings, and lattice tilings. A complete classification of lattice tilings for two certain cases is proved.

1. Introduction

In some applications, information is encoded as a vector of integers, $\mathbf{x} \in \mathbb{Z}^n$, most notably, flash memories (e.g., see [2]). Additionally, a common noise affecting these applications is a limited-magnitude error affecting some of the entries. Namely, at most t entries are increased by as much as k_+ or decreased by as much as k_-. Thus, for integers $n \geqslant t \geqslant 1$, and $k_+ \geqslant k_- \geqslant 0$, we define the (n, t, k_+, k_-)-*error-ball* as

$$\mathcal{B}(n, t, k_+, k_-) \triangleq \left\{ \mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{Z}^n \ : \ -k_- \leqslant x_i \leqslant k_+ \text{ and } wt(\mathbf{x}) \leqslant t \right\},$$

where $wt(\mathbf{x})$ denotes the Hamming weight of $\mathbf{x}$. It now follows that an error-correcting code in this setting is equivalent to a packing of $\mathbb{Z}^n$ by $\mathcal{B}(n, t, k_+, k_-)$, and the subject of interest for this paper, a perfect code is equivalent to a tiling of $\mathbb{Z}^n$ by $\mathcal{B}(n, t, k_+, k_-)$. An example of $\mathcal{B}(3, 2, 2, 1)$ is shown in Fig. 1.

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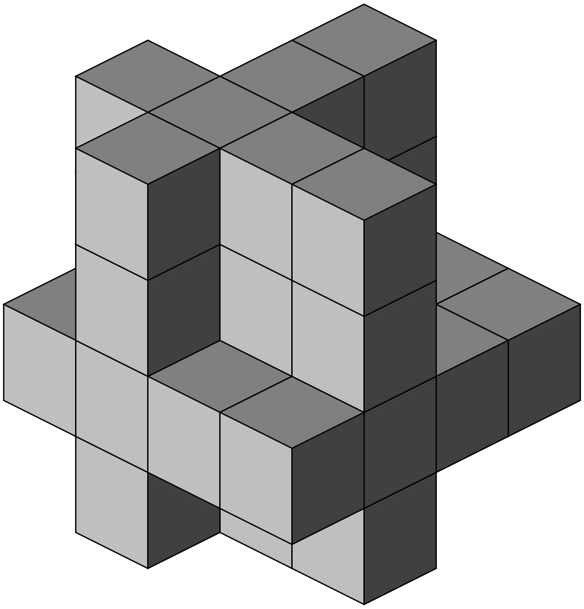

Fig. 1. A depiction of $\mathcal{B}(3, 2, 2, 1)$ where each point in $\mathcal{B}(3, 2, 2, 1)$ is shown as a unit cube.

Previous works on tiling these shapes almost exclusively studied the case of $t = 1$. The *cross*, $\mathcal{B}(n, 1, k, k)$, and semi-cross, $\mathcal{B}(n, 1, k, 0)$ have been extensively researched, e.g., see [3–5,10,12] and the many references therein. This was recently extended to *quasi-crosses*, $\mathcal{B}(n, 1, k_+, k_-)$, in [7], creating a flurry of activity on the subject [8,15–19]. To the best of our knowledge, [11] and later [1], are the only works to consider $t \geqslant 2$, by considering a notched cube (or a "chair"), which for certain parameters becomes $\mathcal{B}(n, n-1, k, 0)$. Tilings of these shapes have been constructed in [1,11]. Additionally, [1] showed that $\mathcal{B}(n, n-2, k, 0)$, $n \geqslant 4$, $k \geqslant 1$, can never lattice-tile $\mathbb{Z}^n$.

The goal of this paper is to study tilings of $\mathcal{B}(n, t, k_+, k_-)$ for $t \geqslant 2$. We first propose a construction of lattice tilings from perfect codes in the Hamming metric, and show that $\mathcal{B}(n, t, k_+, k_-)$ can lattice-tile $\mathbb{Z}^n$ for some special values of n, t, k_+, and k_-, namely, $n = 2t + 1$ and $(k_+, k_-) = (1, 0)$, or $(n, t, k_+, k_-) \in \{(23, 3, 1, 0), (11, 2, 2, 0), (11, 2, 1, 1)\}$. Then we provide a sequence of non-existence results in various parameter regimes, both for lattice tilings and for general non-lattice tilings. Among others, we show that if $2t \geqslant n > t$ and $k_+ > k_- > 0$, or $n > t \geqslant (4n-2)/5$ and $k_+ = k_- \geqslant 2$, then $\mathbb{Z}^n$ cannot be tiled by $\mathcal{B}(n, t, k_+, k_-)$. Furthermore, we show that if $\frac{2}{3}(n-1) \leqslant t \leqslant n-3$, then $\mathcal{B}(n, t, k_+, 0)$ cannot lattice-tile $\mathbb{Z}^n$ when $k_+ \geqslant 2$. Additionally, we provide a complete classification of lattice tilings with $\mathcal{B}(n, 2, 1, 0)$ and $\mathcal{B}(n, 2, 2, 0)$. Our approaches use both algebraic techniques and geometric ones.

The paper is organized as follows: In Section 2 we provide the notation used throughout the paper, as well as definitions and basic results concerning lattice tilings and group splittings. We construct lattice tilings in Section 3, and prove non-existence results in Section 4. We summarize our non-existence results and raise some open questions in Section 5.

2. Preliminaries

Throughout the paper we let n and t be integers such that $n \geqslant t \geqslant 1$. We further assume k_+ and k_- are non-negative integers such that $k_+ \geqslant k_- \geqslant 0$. For integers $a \leqslant b$ we define $[a, b] \triangleq \{a, a+1, \ldots, b\}$ and $[a, b]^* \triangleq [a, b] \setminus \{0\}$. We use $\mathbb{Z}_m$ to denote the cyclic group of integers with addition modulo m, and $\mathbb{F}_q$ to denote the finite field of size q. Since we shall almost always use just the additive group of the finite field, when p is a prime we shall sometimes write $\mathbb{F}_p$ and sometimes $\mathbb{Z}_p$.

A *lattice* $\Lambda \subseteq \mathbb{Z}^n$ is an additive subgroup of $\mathbb{Z}^n$. A lattice Λ may be represented by a matrix $\mathcal{G}(\Lambda) \in \mathbb{Z}^{n\times n}$, the span of whose rows (with integer coefficients) is Λ. A *fundamental region* of Λ is defined as

$$\left\{ \sum_{i=1}^{n} c_i \mathbf{v}_i \ : \ c_i \in \mathbb{R}, 0 \leqslant c_i < 1 \right\},$$

where $\mathbf{v}_i$ is the ith row of $\mathcal{G}(\Lambda)$. It is well known that the volume of the fundamental region is $|\det(\mathcal{G}(\Lambda))|$, and is independent of the choice of $\mathcal{G}(\Lambda)$.

We say $\mathcal{B} \subseteq \mathbb{Z}^n$ *packs* $\mathbb{Z}^n$ by $\Lambda \subseteq \mathbb{Z}^n$, if the translates of $\mathcal{B}$ by elements from Λ do not intersect, namely, for all $\mathbf{v}, \mathbf{v}' \in \Lambda$, $\mathbf{v} \neq \mathbf{v}'$,

$$(\mathbf{v} + \mathcal{B}) \cap (\mathbf{v}' + \mathcal{B}) = \varnothing.$$

We say $\mathcal{B}$ *covers* $\mathbb{Z}^n$ by Λ if

$$\bigcup_{\mathbf{v}\in\Lambda} (\mathbf{v} + \mathcal{B}) = \mathbb{Z}^n.$$

If $\mathcal{B}$ both packs and covers $\mathbb{Z}^n$ by Λ, then we say $\mathcal{B}$ *tiles* $\mathbb{Z}^n$ by Λ. It is well known that if $\mathcal{B}$ packs $\mathbb{Z}^n$ by Λ, and $|\mathcal{B}| = |\det(\mathcal{G}(\Lambda))|$, then $\mathcal{B}$ tiles $\mathbb{Z}^n$ by Λ.

2.1. Lattice tiling and group splitting

Lattice tiling of $\mathbb{Z}^n$ with $\mathcal{B}(n, t, k_+, k_-)$, in connection with group splitting, has a long history when $t = 1$ (e.g., see [9]), called lattice tiling by crosses if $k_+ = k_-$ (e.g., [10]), semi-crosses when $k_- = 0$ (e.g., [3,4,10]), and quasi-crosses when $k_+ \geqslant k_- \geqslant 0$ (e.g., [7,8]). For an excellent treatment and history, the reader is referred to [12] and the many references therein. Other variations, keeping $t = 1$ include [13,14]. More recent results may be found in [16] and the references therein.

Since we are interested in codes that correct more than one error, namely, $t \geqslant 2$, an extended definition of group splitting is required.

Definition 1. Let G be a finite Abelian group, where $+$ denotes the group operation. For $m \in \mathbb{Z}$ and $g \in G$, let mg denote $g + g + \cdots + g$ (with m copies of g) when $m > 0$, which is extended in the natural way to $m \leqslant 0$. Let $M \subseteq \mathbb{Z} \setminus \{0\}$ be a finite set, and $S = \{s_1, s_2, \ldots, s_n\} \subseteq G$. We say the set M t-splits G with splitter set S, denoted

$$G = M \diamond_t S$$

if the following two conditions hold:

1. The elements $\mathbf{e} \cdot (s_1, \ldots, s_n)$, where $\mathbf{e} \in (M \cup \{0\})^n$ and $1 \leqslant wt(\mathbf{e}) \leqslant t$, are all distinct and non-zero in G.
2. For every $g \in G$ there exists a vector $\mathbf{e} \in (M \cup \{0\})^n$ with $wt(\mathbf{e}) \leqslant t$, such that $g = \mathbf{e} \cdot (s_1, \ldots, s_n)$.

Intuitively, $G = M \diamond_t S$ means that the non-trivial linear combinations of elements from S, with at most t non-zero coefficients from M, are distinct and give all the non-zero elements of G exactly once. We note that when $t = 1$, this definition coincides with the definition of splitting used in previous papers.

The following two theorems show the equivalence of t-splittings and lattice tilings, summarizing Lemma 3, Lemma 4, and Corollary 1 in [1]. They generalize the treatment for $t = 1$ in previous works (e.g., see [12]).

Theorem 1 (*Lemma 4 and Corollary 1 in [1]*). *Let G be a finite Abelian group, $M \triangleq [-k_-, k_+]^*$, and $S = \{s_1, \ldots, s_n\} \subseteq G$, such that $G = M \diamond_t S$. Define $\phi : \mathbb{Z}^n \to G$ as $\phi(\mathbf{x}) \triangleq \mathbf{x} \cdot (s_1, \ldots, s_n)$ and let $\Lambda \triangleq \ker \phi$ be a lattice. Then $\mathcal{B}(n, t, k_+, k_-)$ tiles $\mathbb{Z}^n$ by Λ.*

Theorem 2 (*Lemma 3 and Corollary 1 in [1]*). *Let $\Lambda \subseteq \mathbb{Z}^n$ be a lattice, and assume $\mathcal{B}(n, t, k_+, k_-)$ tiles $\mathbb{Z}^n$ by Λ. Then there exist a finite Abelian group G and $S = \{s_1, s_2, \dots, s_n\} \subseteq G$ such that $G = M \diamond_t S$, where $M \triangleq [-k_-, k_+]^*$.*

3. Construction of lattice tilings

In this section we describe a construction for tilings with $\mathcal{B}(n, t, k_+, k_-)$. The method described here takes a linear perfect code in the well known and extensively studied Hamming metric, and uses it to construct the tiling. The obvious downside to this method is the fact that very few perfect codes exist in the Hamming metric (see [6] for more on perfect codes).

Theorem 3. *In the Hamming metric space, let C be a perfect linear $[n, k, 2t + 1]$ code over $\mathbb{F}_p$, with p a prime. If $k_+ + k_- + 1 = p$, then*

$$\Lambda \triangleq \left\{\mathbf{x} \in \mathbb{Z}^n \ : \ (\mathbf{x} \bmod p) \in C\right\}$$

is a lattice, and $\mathcal{B}(n, t, k_+, k_-)$ lattice-tiles $\mathbb{Z}^n$ by Λ.

Proof. Directly from its definition, Λ is closed under addition and under multiplication by integers. Thus, Λ is a lattice. Denote $\mathcal{B} \triangleq \mathcal{B}(n, t, k_+, k_-)$, and we now prove $\mathcal{B}$ tiles $\mathbb{Z}^n$ by Λ.

To show packing, assume $\mathbf{v} + \mathbf{e} = \mathbf{v}' + \mathbf{e}'$, for some $\mathbf{v}, \mathbf{v}' \in \Lambda$ and $\mathbf{e}, \mathbf{e}' \in \mathcal{B}$. But then $\mathbf{e} - \mathbf{e}' = \mathbf{v}' - \mathbf{v} \in \Lambda$, and by the definition of Λ, also $\mathbf{e}'' \triangleq ((\mathbf{e} - \mathbf{e}') \bmod p) \in C$. We note that $wt(\mathbf{e}) \leqslant t$ and $wt(\mathbf{e}') \leqslant t$, hence $wt(\mathbf{e}'') \leqslant 2t$. By the minimum distance of C this implies that $\mathbf{e}'' = \mathbf{0}$. Now, since each entry of $\mathbf{e} - \mathbf{e}'$ is in the range $[-(k_+ + k_-), k_+ + k_-]$, and since $k_+ + k_- + 1 = p$, we necessarily have that $\mathbf{e} - \mathbf{e}' = \mathbf{0}$, which in turn implies $\mathbf{v} - \mathbf{v}' = \mathbf{0}$. It follows that translates of $\mathcal{B}$ by Λ pack $\mathbb{Z}^n$.

To show covering, let $\mathbf{x} \in \mathbb{Z}^n$ be any integer vector. Then $\mathbf{x}' \triangleq (\mathbf{x} \bmod p) \in \mathbb{F}_p^n$. Since C is a perfect code, there exist $\mathbf{v}' \in C$ and $\mathbf{e}' \in \mathbb{F}_p^n$, $wt(\mathbf{e}') \leqslant t$, such that $\mathbf{x}' \equiv \mathbf{v}' + \mathbf{e}' \pmod p$. Since $k_+ + k_- + 1 = p$, there exists $\mathbf{e} \in \mathcal{B}$ such that $\mathbf{e} \bmod p = \mathbf{e}'$. But then $\mathbf{x} - \mathbf{e} \equiv \mathbf{v}' \pmod p$ and by definition $\mathbf{x} - \mathbf{e} \in \Lambda$. Hence, the translates of $\mathcal{B}$ by Λ cover $\mathbb{Z}^n$. $\square$

Example 1. Take the $[\frac{p^m-1}{p-1}, \frac{p^m-1}{p-1} - m, 3]$ p-ary Hamming code (p a prime), together with Theorem 3, to obtain a tiling of $\mathbb{Z}^{(p^m-1)/(p-1)}$ by $\mathcal{B}(\frac{p^m-1}{p-1}, 1, k_+, k_-)$, where $k_+ + k_- + 1 = p$. This particular tiling was already described in [7] together with the lattice generator matrix and equivalent splitting.

Example 2. If we use Theorem 3 with the perfect binary linear $[2t + 1, 1, 2t + 1]$ repetition code, we obtain a lattice tiling of $\mathbb{Z}^{2t+1}$ by $\mathcal{B}(2t + 1, t, 1, 0)$. The lattice is spanned by

$$\mathcal{G} = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ & 2 & & & \\ & & 2 & & \\ & & & \ddots & \\ & & & & 2 \end{pmatrix}.$$

When viewed as a splitting, the additive group $\mathbb{F}_2^{2t}$ is t-split as $\mathbb{F}_2^{2t} = \{1\} \diamond_t S$, where $S = \{\mathbf{e}_i \ : \ 1 \leqslant i \leqslant 2t\} \cup \{\mathbf{1}\}$, and where $\mathbf{e}_i$ is the ith unit vector of length $2t$.

Example 3. Again using Theorem 3 with the $[23, 12, 7]$ binary Golay code, we obtain a lattice tiling of $\mathbb{Z}^{23}$ by $\mathcal{B}(23, 3, 1, 0)$. The lattice Λ is spanned by

$$\mathcal{G} = \begin{pmatrix} I_{12} & G_b \\ \mathbf{0} & 2I_{11} \end{pmatrix},$$

where $\begin{pmatrix} I_{12} & G_b \end{pmatrix}$ is a generator matrix of the [23, 12, 7] binary Golay code, and $2I_{11}$ is an 11×11 matrix with entries on the diagonal being 2 and all the others being 0. Now, we look at the corresponding group splitting. Since $\mathbb{Z}^{23}$ can be spanned by the matrix

$$\begin{pmatrix} I_{12} & G_b \\ \mathbf{0} & I_{11} \end{pmatrix},$$

the quotient group $\mathbb{Z}^{23}/\Lambda$ is isomorphic to the additive group $\mathbb{F}_2^{11}$. Note that

$$\begin{pmatrix} I_{12} & G_b \\ \mathbf{0} & 2I_{11} \end{pmatrix} \begin{pmatrix} G_b \\ I_{11} \end{pmatrix}$$

is a 23×11 all-zero matrix over $\mathbb{F}_2$. The natural homomorphism $\phi : \mathbb{Z}^{23} \to \mathbb{F}_2^{11}$ sends the standard basis to the rows of $\begin{pmatrix} G_b \\ I_{11} \end{pmatrix}$. It follows that $\mathbb{F}_2^{11} = \{1\} \diamond_3 S$, where $S = \{\mathbf{e}_i : 1 \leqslant i \leqslant 11\} \cup \{\mathbf{r} : \mathbf{r} \text{ is a row of } G_b\}$.

Example 4. Finally, using Theorem 3 with the [11, 6, 5] ternary Golay code, we obtain a lattice tiling of $\mathbb{Z}^{11}$ by $\mathcal{B}(11, 2, 2, 0)$ or $\mathcal{B}(11, 2, 1, 1)$. The lattice is spanned by

$$\mathcal{G} = \begin{pmatrix} I_6 & G_t \\ \mathbf{0} & 3I_5 \end{pmatrix},$$

where $\begin{pmatrix} I_6 & G_t \end{pmatrix}$ is a generator matrix of the [11, 6, 5] ternary Golay code, and $3I_5$ is a 5×5 matrix with entries on the diagonal being 3 and all the others being 0. When viewed as a splitting, the additive group $\mathbb{F}_3^5$ is 2-split as $\mathbb{F}_3^5 = \{1, 2\} \diamond_2 S$, where $S = \{\mathbf{e}_i : 1 \leqslant i \leqslant 5\} \cup \{\mathbf{r} : \mathbf{r} \text{ is a row of } G_t\}$.

Theorem 3 has its dual as well, as shown in the following theorem.

Theorem 4. *Assume $\mathcal{B}(n, t, k_+, k_-)$ lattice-tiles $\mathbb{Z}^n$ by the lattice Λ, with an equivalent t-splitting $\mathbb{F}_p^m = M \diamond_t S$, where $M \triangleq [-k_-, k_+]^*$, p is a prime, and $p = k_+ + k_- + 1$. Then $\Lambda \cap \mathbb{F}_p^n$ is a perfect linear $[n, k, 2t + 1]$ code over $\mathbb{F}_p$ in the Hamming metric space.*

Proof. By Theorems 1 and 2, $\Lambda = \ker\phi$, where $\phi : \mathbb{Z}^n \to \mathbb{F}_p^m$, with $S = \{s_1, \dots, s_n\} \subseteq \mathbb{F}_p^m$, and $\phi(\mathbf{x}) = \mathbf{x} \cdot (s_1, \dots, s_n)$. Let $\mathbf{e}_i \in \mathbb{Z}^n$ be the ith standard unit vector. Due to the characteristic of $\mathbb{F}_p^n$, for all $\mathbf{x} \in \mathbb{Z}^n$, $\phi(\mathbf{x}) = \phi(\mathbf{x} + p\mathbf{e}_i)$. It follows that

$$\Lambda = \Lambda + p\mathbf{e}_i, \tag{1}$$

for all $i = 1, 2, \dots, n$. In turn, this implies that

$$\Lambda \cap \mathbb{F}_p^n = \Lambda \bmod p \triangleq \{\mathbf{x} \bmod p : \mathbf{x} \in \Lambda\}. \tag{2}$$

Since Λ is a lattice, we then have that $C \triangleq \Lambda \cap \mathbb{F}_p^n$ is a vector space, namely, a linear code.

It remains to show C is a perfect code with the claimed parameters. Let $\mathbf{c}, \mathbf{c}' \in C$ be two distinct codewords, and $\mathbf{e}, \mathbf{e}' \in \mathbb{F}_p^n$ be two error patterns, $wt(\mathbf{e}), wt(\mathbf{e}') \leqslant t$. Assume to the contrary that

$$\mathbf{c} + \mathbf{e} \equiv \mathbf{c}' + \mathbf{e}' \pmod{p},$$

where we emphasize that addition here is in $\mathbb{F}_p^n$ by writing that the equivalence holds modulo p. Since $k_+ + k_- + 1 = p$, there are unique vectors $\mathbf{f}, \mathbf{f}' \in \mathcal{B}(n, t, k_+, k_-)$ such that

$$\mathbf{f} \equiv \mathbf{e} \pmod{p} \qquad \text{and} \qquad \mathbf{f}' \equiv \mathbf{e}' \pmod{p}.$$

We now have

$$\mathbf{c} + \mathbf{f} \equiv \mathbf{c}' + \mathbf{f}' \pmod{p},$$

hence there exists $\mathbf{v} \in \mathbb{Z}^n$ such that

$$\mathbf{c} + \mathbf{f} = \mathbf{c}' + \mathbf{f}' + p\mathbf{v}.$$

If we define $\mathbf{c}'' = \mathbf{c}' + p\mathbf{v}$, then by (1), $\mathbf{c}'' \in \Lambda$. But then

$$\mathbf{c} + \mathbf{f} = \mathbf{c}'' + \mathbf{f}',$$

contradicting the fact that $\mathcal{B}(n, t, k_+, k_-)$ tiles $\mathbb{Z}^n$ by Λ. Thus, C is a linear $[n, k, \geqslant 2t + 1]$ code over $\mathbb{F}_p$.

Finally, we show C is perfect. Let $\mathbf{u} \in \mathbb{F}_p^n$ be any vector. Since $\mathcal{B}(n, t, k_+, k_-)$ tiles $\mathbb{Z}^n$ by Λ, there exist $\mathbf{v} \in \Lambda$ and $\mathbf{e} \in \mathcal{B}(n, t, k_+, k_-)$ such that $\mathbf{u} = \mathbf{v} + \mathbf{e}$. Taking the equation modulo p, we get that

$$\mathbf{u} \equiv \mathbf{v} + \mathbf{e} \pmod p,$$

where we emphasize that $\mathbf{u} \bmod p = \mathbf{u}$. By (2), $\mathbf{v} \bmod p \in C$. Additionally, since $k_+ + k_- + 1 = p$, we have that $wt(\mathbf{e}) = wt(\mathbf{e} \bmod p) \leqslant t$. Thus C has covering radius at most t, and it is therefore a perfect code, as claimed. □

4. Nonexistence results

The nonexistence results we present in this section are divided into results on general tilings, and results on lattice tilings. The former use mainly geometric arguments, whereas the latter employ algebraic ones.

4.1. Nonexistence of general tilings

The first result we present uses a comparison between the density of a tiling of $\mathcal{B}(n, t, k_+, k_-)$ with that of a tiling of a certain notched cube of a lower dimension.

Theorem 5. *For any $n \geqslant t + 1$, and $k_+ \geqslant k_- \geqslant 0$ not both 0, if*

$$\sum_{i=0}^{t} \binom{n}{i} (k_+ + k_-)^i < (k_+ + 1)^{t+1} - (k_+ - k_-)^{t+1}$$

then $\mathbb{Z}^n$ cannot be tiled by translates of $\mathcal{B}(n, t, k_+, k_-)$.

Proof. Given integers $n \geqslant t + 1$, assume that there is a set $T \subseteq \mathbb{Z}^n$ such that $\mathcal{B} \triangleq \mathcal{B}(n, t, k_+, k_-)$ tiles $\mathbb{Z}^n$ by T. Consider the set

$$A = \left\{ (x_1, x_2, \dots, x_{t+1}, 0, \dots, 0) \ : \ (x_1, \dots, x_{t+1}) \in [0, k_+]^{t+1} \setminus [k_- + 1, k_+]^{t+1} \right\}.$$

Hence, if we remove the last $n - t - 1$ zero coordinates, the elements of A are exactly a notched cube, as defined in [1,11]. Thus, by [1,11], translates of A tile the space[1]

$$\{ (x_1, x_2, \dots, x_{t+1}, 0, \dots, 0) \ : \ x_i \in \mathbb{Z} \text{ for all } 1 \leqslant i \leqslant t + 1 \}.$$

Trivially, it follows that translates of A can tile the space $\mathbb{Z}^n$.

We now claim that any translate of A contains at most one point from T. Suppose to the contrary that both $\mathbf{x} = (x_1, x_2, \dots, x_n)$ and $\mathbf{y} = (y_1, y_2, \dots, y_n)$ belong to the intersection $(\mathbf{v} + A) \cap T$, where $\mathbf{v} = (v_1, v_2, \dots, v_n) \in \mathbb{Z}^n$, and $\mathbf{x} \neq \mathbf{y}$. Then $v_i \leqslant x_i, y_i \leqslant v_i + k_+$ for $1 \leqslant i \leqslant t + 1$, $x_i = y_i = v_i$ for $t + 2 \leqslant i \leqslant n$, and there are indices $1 \leqslant j_x, j_y \leqslant t + 1$ such that $x_{j_x} \leqslant v_{j_x} + k_-$ and $y_{j_y} \leqslant v_{j_y} + k_-$. W.l.o.g., assume that $x_1 \leqslant v_1 + k_-$. We proceed in two cases.

(1) If $y_1 \leqslant v_1 + k_-$, let $\mathbf{z} = (z_1, z_2, \dots, z_{t+1}, v_{t+2}, v_{t+3} \dots, v_n)$, where

$$z_1 = \begin{cases} x_1, & \text{if } x_i \leqslant y_i \text{ for all } i = 2, 3, \dots, t + 1, \\ y_1, & \text{otherwise,} \end{cases}$$

[1] While [1,11] discuss a tiling of $\mathbb{R}^n$, it is easily seen that the tiling constructed there is in fact a tiling of $\mathbb{Z}^n$ as in our setting.

and

$$z_i = \max\{x_i, y_i\} \text{ for } i = 2, 3, \dots, t+1.$$

Then it is easy to see that

$$\mathbf{z} \in (\mathbf{x} + \mathcal{B}) \cap (\mathbf{y} + \mathcal{B}),$$

a contradiction.

(2) If $y_1 > v_1 + k_-$, then there is $2 \leqslant j \leqslant t+1$ such that $y_j \leqslant v_j + k_-$. W.l.o.g., assume that $y_2 \leqslant v_2 + k_-$ and let $\mathbf{z} = (y_1, z_2, z_3, \dots, z_{t+1}, v_{t+2}, v_{t+3} \dots, v_n)$, where

$$z_2 = \begin{cases} x_2, & \text{if } x_i \leqslant y_i \text{ for all } i = 2, 3, \dots, t+1, \\ \max\{x_2, y_2\}, & \text{otherwise}, \end{cases}$$

and

$$z_i = \max\{x_i, y_i\} \text{ for } i = 3, 4, \dots, t+1.$$

Again,

$$\mathbf{z} \in (\mathbf{x} + \mathcal{B}) \cap (\mathbf{y} + \mathcal{B}),$$

a contradiction.

We have shown that any translate of A contains at most one point from T, and so the tiling by A is denser than the tiling by $\mathcal{B}$. It follows that the reciprocal of the volume of $\mathcal{B}$ cannot exceed the reciprocal of the volume of A, i.e.,

$$\frac{1}{\sum_{i=0}^{t} \binom{n}{i}(k_+ + k_-)^i} \leqslant \frac{1}{(k_+ + 1)^{t+1} - (k_+ - k_-)^{t+1}}.$$

Rearranging gives us the desired result. $\square$

Remark 1. If $k_- \geqslant ck_+$ for some real number $c > 0$, while n and t are fixed, then according to Theorem 5, there is an upper bound on k_+ for which $\mathcal{B}(n, t, k_+, k_-)$ can tile $\mathbb{Z}^n$.

Next, we study a case which is analogous to that of proper quasi-crosses when $t = 1$, namely, the case when $k_+ > k_- > 0$. The main tool is a geometric one, studying the two translates of $\mathcal{B}(n, t, k_+, k_-)$ that cover the all-zero and all-one vectors.

Theorem 6. *Let $2t \geqslant n \geqslant t+1$ and $k_+ > k_- > 0$. Then $\mathbb{Z}^n$ cannot be tiled by $\mathcal{B}(n, t, k_+, k_-)$.*

Proof. Denote $\mathcal{B} \triangleq \mathcal{B}(n, t, k_+, k_-)$, and assume to the contrary that there is a set $T \subseteq \mathbb{Z}^n$ such that $\mathcal{B}$ tiles $\mathbb{Z}^n$ by T. W.l.o.g., we may assume that the all-zero vector $\mathbf{0}$ is in T.

We consider the all-one vector $\mathbf{1}$. Since $\mathbf{1} \notin \mathcal{B}$, there is a non-zero vector $\mathbf{a} = (a_1, a_2, \dots, a_n) \in T$ such that $\mathbf{1} \in \mathbf{a} + \mathcal{B}$, where $1 - k_+ \leqslant a_i \leqslant 1 + k_-$ for $1 \leqslant i \leqslant n$. By interchanging the coordinates, we may assume, w.l.o.g., that

$$a_i = 1 \text{ for } 1 \leqslant i \leqslant n-t, \text{ and } a_i \geqslant a_{i+1} \text{ for } n-t+1 \leqslant i \leqslant n-1.$$

If $a_{t+1} < 1 + k_-$, then $1 - k_+ \leqslant a_i \leqslant k_-$ for $t+1 \leqslant i \leqslant n$. Since by assumption $n - t \leqslant t$, it follows that

$$(\underbrace{1, 1, \dots, 1}_{t}, \underbrace{0, 0, \dots, 0}_{n-t}) \in (\mathbf{a} + \mathcal{B}) \cap (\mathbf{0} + \mathcal{B}),$$

which contradicts the assumption that $\mathcal{B}$ tiles $\mathbb{Z}^n$ by T. Hence, $a_{t+1} = 1 + k_-$.

Now, let i_0 be the largest index such that $a_{i_0} = 1 + k_-$. Then $i_0 - t \geqslant 1$ as $a_{t+1} = 1 + k_-$. Consider the vector

$$\mathbf{v} \triangleq (\underbrace{1, 1, \dots, 1}_{n-i_0}, \underbrace{0, 0, \dots, 0}_{i_0-t}, a_{n-t+1}, a_{n-t+2}, \dots, a_{i_0}, \underbrace{0, 0, \dots, 0}_{n-i_0}).$$

We first compare $\mathbf{v}$ with $\mathbf{a}$. Note that $(n-i_0)+(i_0-t)=n-t$ and $a_i=1$ for $1\leqslant i\leqslant n-t$. Hence, $\mathbf{v}$ can be obtained from $\mathbf{a}$ by changing $n-t$ a_i's to 0, i.e., those a_i's with $n-i_0+1\leqslant i\leqslant n-t$ or $i_0+1\leqslant i\leqslant n$. Since $n-t\leqslant t$, $a_i=1\leqslant k_-$ for $n-i_0+1\leqslant i\leqslant n-t$, and $1-k_+\leqslant a_i\leqslant k_-$ for $i_0+1\leqslant i\leqslant n$, we have $\mathbf{v}\in\mathbf{a}+\mathcal{B}$.

Second, we compare $\mathbf{v}$ with $\mathbf{0}$. Note that $(i_0-t)+(n-i_0)=n-t$. These two vectors differ in at most t positions. Hence, $\mathbf{v}$ can be obtained from $\mathbf{0}$ by changing the first $n-i_0$ 0's to 1 and the ith 0 to a_i for $n-t+1\leqslant i\leqslant i_0$. Since $-k_-\leqslant a_i\leqslant 1+k_-$ for $n-t+1\leqslant i\leqslant i_0$, $k_+\geqslant 1$ and $1+k_-\leqslant k_+$, we have that $\mathbf{v}\in\mathbf{0}+\mathcal{B}$.

It follows that

$$\mathbf{v}\in(\mathbf{a}+\mathcal{B})\cap(\mathbf{0}+\mathcal{B}),$$

which again contradicts the assumption that $\mathcal{B}$ tiles $\mathbb{Z}^n$ by T. $\square$

For the last result concerning general tiling, we study the case of equal arm length, $k_+=k_-$. The method used is an elaboration of the one used in the proof of Theorem 6: instead of considering only the all-zero and all-one vectors, we consider a third vector as well.

Theorem 7. *Let $k_+=k_-\geqslant 2$ and $n>t\geqslant(4n-2)/5$. Then for any $n\geqslant 3$, $\mathbb{Z}^n$ cannot be tiled by $\mathcal{B}(n,t,k_+,k_-)$.*

Proof. Let $k\triangleq k_+=k_-$ and $\tau\triangleq n-t$. Suppose to the contrary that there is a set $T\subseteq\mathbb{Z}^n$ such that $\mathcal{B}\triangleq\mathcal{B}(n,t,k_+,k_-)$ tiles $\mathbb{Z}^n$ by T. W.l.o.g., we assume that $\mathbf{0}\in T$. Since $t\geqslant(4n-2)/5$ and $n\geqslant 3$, we have $t\geqslant n/2$. According to the first three paragraphs in the proof of Theorem 6, we may assume that $\mathbf{1}\in\mathbf{a}+\mathcal{B}$, where

$$\mathbf{a}\triangleq(\underbrace{1,1,\ldots,1}_{\tau},\underbrace{1+k,1+k,\ldots,1+k}_{i_0-\tau},a_{i_0+1},\ldots,a_n)\in T,$$

with $i_0\geqslant t+1$, and $1-k\leqslant a_i\leqslant k$ for $i_0+1\leqslant i\leqslant n$.

We consider the vector

$$\mathbf{v}\triangleq(\underbrace{2,2,\ldots,2}_{\tau},\underbrace{1,1,\ldots,1}_{i_0-\tau},a_{i_0+1},\ldots,a_n).$$

It is not contained in $(\mathbf{0}+\mathcal{B})\cup(\mathbf{a}+\mathcal{B})$ as the Hamming distance between $\mathbf{v}$ and $\mathbf{0}$ or $\mathbf{v}$ and $\mathbf{a}$ is at least $i_0\geqslant t+1$. We assume that $\mathbf{v}$ is contained in another ball centred at $\mathbf{b}=(b_1,b_2,\ldots,b_n)\in T$, where $1-k\leqslant b_i\leqslant 1+k$ for $\tau+1\leqslant i\leqslant i_0$. Let $c\triangleq|\{i\ :\ \tau+1\leqslant i\leqslant i_0, b_i=1+k\}|$. We proceed in the following two cases.

1. If $c\leqslant i_0-3\tau$, by interchanging all the coordinates between $\tau+1$ and i_0, we may assume that $1-k\leqslant b_i\leqslant k$ for $i_0-2\tau+1\leqslant i\leqslant i_0$. We consider the vector

$$\mathbf{x}\triangleq(\underbrace{2,\ldots,2}_{\tau},\underbrace{1,1,\ldots,1}_{i_0-3\tau},\underbrace{0,\ldots,0}_{\tau},\underbrace{b_{i_0-\tau+1}\ldots,b_{i_0}}_{\tau},a_{i_0+1},\ldots,a_n).$$

   We first compare $\mathbf{x}$ with $\mathbf{0}$. These two vectors agree in at least $\tau=n-t$ positions. Noting that $k\geqslant 2$, $1-k\leqslant b_i\leqslant k$ for $i_0-\tau+1\leqslant i\leqslant i_0$ and $1-k\leqslant a_i\leqslant k$ for $i_0+1\leqslant i\leqslant n$, we have $\mathbf{x}\in\mathbf{0}+\mathcal{B}$. Second, we compare $\mathbf{x}$ with $\mathbf{b}$. They differ in the first $i_0-\tau$ positions and the last $n-i_0$ positions, and so in total $n-\tau=t$ positions. Noting that $\mathbf{x}$ and $\mathbf{v}$ agree in the first $i_0-2\tau$ positions and the last $n-i_0$ positions and $\mathbf{v}\in\mathbf{b}+\mathcal{B}$, the symbols of $\mathbf{x}$ in these positions can be obtained from the corresponding symbols of $\mathbf{b}$ by adding or subtracting up to k units. For the remaining τ positions where $i_0-2\tau+1\leqslant i\leqslant i_0-\tau$, we have $1-k\leqslant b_i\leqslant k$. It follows that $\mathbf{x}\in\mathbf{b}+\mathcal{B}$ and then

$$\mathbf{x}\in(\mathbf{b}+\mathcal{B})\cap(\mathbf{0}+\mathcal{B}).$$

2. If $c > i_0 - 3\tau$, we may assume that $b_i = 1 + k$ for $\tau + 1 \leqslant i \leqslant i_0 - 2\tau + 1$. Consider the vector

$$\mathbf{y} \triangleq (\underbrace{2, \ldots, 2}_{\tau}, \underbrace{1+k, \ldots, 1+k}_{i_0-3\tau+1}, 1, \ldots, 1, a_{i_0+1}, \ldots, a_n).$$

We first compare $\mathbf{y}$ with $\mathbf{b}$. These two vectors differ in the first τ positions and the last $n - i_0 + 2\tau - 1$ positions. Since $t \geqslant (4n-2)/5$, they differ in a total of $n - i_0 + 3\tau - 1 \leqslant 4\tau - 2 = 4n - 2 - 4t \leqslant t$ positions. Noting that $\mathbf{y}$ and $\mathbf{v}$ agree in these positions and $\mathbf{v} \in \mathbf{b} + \mathcal{B}$, we have $\mathbf{y} \in \mathbf{b} + \mathcal{B}$. Second, we compare $\mathbf{y}$ with $\mathbf{a}$. They differ in a total of $\tau + (i_0 - \tau) - (i_0 - 3\tau + 1) = 3\tau - 1 = 3n - 3t - 1$ positions. Note that $t \geqslant (4n-2)/5 \geqslant (3n-1)/4$ as $n \geqslant 3$. Thus we have $3n - 3t - 1 \leqslant t$. Furthermore, in these $3n - 3t - 1$ positions, the corresponding symbols differ by at most k units. It follows that $\mathbf{y} \in \mathbf{a} + \mathcal{B}$ and then

$$\mathbf{y} \in (\mathbf{a} + \mathcal{B}) \cap (\mathbf{b} + \mathcal{B}).$$

In both cases above we obtain a contradiction to the assumption that $\mathcal{B}$ tiles $\mathbb{Z}^n$ by T. $\square$

4.2. Nonexistence of lattice tilings

We now turn to the more specific case of lattice tilings. Some of the nonexistence results presented in this section are stated as necessary conditions. The main tool used is Theorem 2, and the algebraic study of the t-splitting. We begin with the lattice-tiling equivalent of Theorem 5.

Theorem 8. *For any $n \geqslant t + 1$, and $k_+ \geqslant k_- \geqslant 0$ not both 0, if $\mathcal{B}(n, t, k_+, k_-)$ lattice-tiles $\mathbb{Z}^n$ then*

$$\sum_{i=1}^{t} \binom{n}{i} (k_+ + k_-)^{i-1} \geqslant (k_- + 1)^t.$$

Proof. For $t = 1$, see [7, Theorem 11]. In the following, we focus on the cases $t \geqslant 2$. Assume that $\mathcal{B} \triangleq \mathcal{B}(n, t, k_+, k_-)$ lattice-tiles $\mathbb{Z}^n$. By Theorem 2 there is an Abelian group G with $|G| = \sum_{i=0}^{t} \binom{n}{i} (k_+ + k_-)^i$ and a subset $S = \{s_1, s_2, \ldots, s_n\} \subseteq G$ such that $G = M \diamond_t S$, where $M \triangleq [-k_-, k_+]^*$.

We first claim that for all $2 \leqslant i_1 < i_2 < \cdots < i_t \leqslant n$ there are integers $x_1^{i_1,i_2,\ldots,i_t}, x_{i_1}^{i_1,i_2,\ldots,i_t}, \ldots, x_{i_t}^{i_1,i_2,\ldots,i_t}$ such that $0 \leqslant x_1^{i_1,i_2,\ldots,i_t} \leqslant \left\lfloor \frac{|G|}{(k_-+1)^t} \right\rfloor$, $|x_{i_j}^{i_1,i_2,\ldots,i_t}| \leqslant k_-$ for $j = 1, 2, \ldots, t$, and

$$s_1 x_1^{i_1,i_2,\ldots,i_t} + s_{i_1} x_{i_1}^{i_1,i_2,\ldots,i_t} + \cdots + s_{i_t} x_{i_t}^{i_1,i_2,\ldots,i_t} = 0.$$

To prove this, fix $i_1, i_2, \ldots, i_t$ and look at the integers $0 \leqslant a_1 \leqslant \left\lfloor \frac{|G|}{(k_-+1)^t} \right\rfloor$, $0 \leqslant a_{i_j} \leqslant k_-$ for $j = 1, 2, \ldots, t$ and the sums $s_1 a_1 + s_{i_1} a_{i_1} + \cdots + s_{i_t} a_{i_t}$. Since

$$\left(\left\lfloor \frac{|G|}{(k_- + 1)^t} \right\rfloor + 1 \right) (k_- + 1)^t \geqslant |G| - ((k_- + 1)^t - 1) + (k_- + 1)^t = |G| + 1 > |G|,$$

by the pigeonhole principle there exist two sequences of integers, $(b_1, b_{i_1}, \ldots, b_{i_t})$ and $(c_1, c_{i_1}, \ldots, c_{i_t})$, such that

$$s_1 b_1 + s_{i_1} b_{i_1} + \cdots + s_{i_t} b_{i_t} = s_1 c_1 + s_{i_1} c_{i_1} + \cdots + s_{i_t} c_{i_t}.$$

Assume, w.l.o.g., that $b_1 \geqslant c_1$ and define $d_1 \triangleq b_1 - c_1$ and $d_{i_j} \triangleq b_{i_j} - c_{i_j}$ for $j = 1, 2, \ldots, t$. We now get

$$s_1 d_1 + s_{i_1} d_{i_1} + \cdots + s_{i_t} d_{i_t} = 0,$$

where $(d_1, d_{i_1}, \ldots, d_{i_t}) \neq (0, 0, \ldots, 0)$. In addition

$$0 \leqslant d_1 \leqslant \left\lfloor \frac{|G|}{(k_- + 1)^t} \right\rfloor \text{ and } |d_{i_j}| \leqslant k_- \text{ for } j = 1, 2, \ldots, t,$$

which prove our claim.

Now, if $\left\lfloor \frac{|G|}{(k_-+1)^t} \right\rfloor \leqslant k_+ + k_-$, then $0 \leqslant d_1 \leqslant \left\lfloor \frac{|G|}{(k_-+1)^t} \right\rfloor \leqslant k_+ + k_-$, and

$$s_1 k_+ + s_{i_1} d_{i_1} + \cdots + s_{i_{t-1}} d_{i_{t-1}} = s_1(k_+ - d_1) - s_{i_t} d_{i_t},$$

which contradicts the fact that $G = M \diamond_t S$, since $t \geqslant 2$.

Hence, we have that

$$k_+ + k_- + 1 \leqslant \left\lfloor \frac{|G|}{(k_- + 1)^t} \right\rfloor.$$

It follows that

$$(k_- + 1)^t \leqslant \frac{|G|}{k_+ + k_- + 1} < \frac{\sum_{i=0}^{t} \binom{n}{i}(k_+ + k_-)^i}{k_+ + k_-} = \sum_{i=1}^{t} \binom{n}{i}(k_+ + k_-)^{i-1} + \frac{1}{k_+ + k_-}.$$

Since both $(k_- + 1)^t$ and $\sum_{i=1}^{t} \binom{n}{i}(k_+ + k_-)^{i-1}$ are integers and $\frac{1}{k_++k_-}$ is at most 1, we have

$$(k_- + 1)^t \leqslant \sum_{i=1}^{t} \binom{n}{i}(k_+ + k_-)^{i-1}. \quad \square$$

Using similar arguments to the previous theorem, the next one specializes in the case of $n \geqslant 2t$.

Theorem 9. *Let $n \geqslant 2t$, and $k_+ \geqslant k_- \geqslant 0$. If $\mathcal{B}(n, t, k_+, k_-)$ lattice-tiles $\mathbb{Z}^n$ then*

$$\frac{(k_- + 1)^2}{k_+ + k_- + 1} < \binom{n}{t}^{1/t}.$$

Proof. If $\mathcal{B}(n, t, k_+, k_-)$ lattice-tiles $\mathbb{Z}^n$, by Theorem 2 there is an Abelian group G with $|G| = \sum_{i=0}^{t} \binom{n}{i}(k_++k_-)^i$ and a subset $S = \{s_1, s_2, \ldots, s_n\} \subseteq G$ such that $G = M \diamond_t S$, where $M \triangleq [-k_-, k_+]^*$. We consider the sums

$$x_1 s_1 + x_2 s_2 + \cdots + x_t s_t + y_1 s_{t+1} + y_2 s_{t+2} + \cdots + y_t s_{2t},$$

where $0 \leqslant x_i < \frac{k_++k_-+1}{k_-+1} \binom{n}{t}^{1/t}$ and $0 \leqslant y_i \leqslant k_-$ for $i = 1, 2, \ldots, t$. The total number of such sums is at least $\binom{n}{t}(k_+ + k_- + 1)^t$. Noting that

$$\binom{n}{t}(k_+ + k_- + 1)^t = \sum_{i=0}^{t} \binom{n}{t}\binom{t}{i}(k_+ + k_-)^i > \sum_{i=0}^{t} \binom{n}{i}(k_+ + k_-)^i = |G|,$$

there are two sums which are equal. Namely, there are

$$\mathbf{a}, \mathbf{a}' \in \left[0, \left\lceil \frac{k_+ + k_- + 1}{k_- + 1} \binom{n}{t}^{1/t} \right\rceil - 1\right]^t \quad \text{and} \quad \mathbf{b}, \mathbf{b}' \in [0, k_-]^t,$$

with $(\mathbf{a}, \mathbf{b}) \neq (\mathbf{a}', \mathbf{b}')$, such that

$$\mathbf{a} \cdot (s_1, s_2, \ldots, s_t) + \mathbf{b} \cdot (s_{t+1}, s_{t+2}, \ldots, s_{2t}) = \mathbf{a}' \cdot (s_1, s_2, \ldots, s_t) + \mathbf{b}' \cdot (s_{t+1}, s_{t+2}, \ldots, s_{2t}).$$

Let $\mathbf{c} = \mathbf{a} - \mathbf{a}'$ and $\mathbf{d} = \mathbf{b}' - \mathbf{b}$. Rearranging the terms, we have

$$\mathbf{c} \cdot (s_1, s_2, \ldots, s_t) = \mathbf{d} \cdot (s_{t+1}, s_{t+2}, \ldots, s_{2t}).$$

Since $\mathbf{c} \in \left[-\left\lceil \frac{k_++k_-+1}{k_-+1} \binom{n}{t}^{1/t} \right\rceil + 1, \left\lceil \frac{k_++k_-+1}{k_-+1} \binom{n}{t}^{1/t} \right\rceil - 1\right]^t$, $\mathbf{d} \in [-k_-, k_-]^t$, and $(\mathbf{c}, \mathbf{d}) \neq (\mathbf{0}, \mathbf{0})$, to avoid contradicting the assumption $G = M \diamond_t S$, necessarily

$$k_- < \left\lceil \frac{k_+ + k_- + 1}{k_- + 1} \binom{n}{t}^{1/t} \right\rceil - 1,$$

which implies

$$k_- < \frac{k_+ + k_- + 1}{k_- + 1}\binom{n}{t}^{1/t} - 1.$$

The claim now follows by rearranging. □

Theorem 9 is particularly useful in an asymptotic regime where $t = \Theta(n)$, as shown in the following corollary.

Corollary 1. *If $\alpha \leqslant \frac{t}{n} \leqslant \frac{1}{2}$, $k_+ \geqslant k_- \geqslant 0$, and*

$$\frac{(k_- + 1)^2}{k_+ + k_- + 1} \geqslant \frac{e}{\alpha},$$

then $\mathcal{B}(n, t, k_+, k_-)$ does not lattice-tile $\mathbb{Z}^n$.

Proof. We observe that

$$\frac{(k_- + 1)^2}{k_+ + k_- + 1} \geqslant \frac{e}{\alpha} \geqslant \frac{ne}{t} > \binom{n}{t}^{1/t},$$

and the claim now follows by Theorem 9. □

We continue on to a few more specific cases. The next two theorems deal with the analogue of semi-crosses when $t = 1$, namely, the case of $k_- = 0$. First a technical lemma is required.

Lemma 1. *Let $A \subseteq [0, \binom{n}{t} - 1]$ be a subset of size $\binom{n-1}{t}$. If*

$$\left(\frac{n}{4t} - 1\right)\binom{n-1}{t-1} > \frac{1}{2},$$

then A contains two elements a and b such that $b = 2a \neq 0$.

Proof. Define

$$m \triangleq \left\lfloor \frac{1}{2}\left(\binom{n}{t} - 1\right)\right\rfloor,$$

and

$$B \triangleq \bigcup_{i=1}^{m} \{i, 2i\}.$$

Then B is a subset of $[0, \binom{n}{t} - 1]$ with $|B| = 2m - \lfloor m/2 \rfloor$. Consider the intersection of A and B,

$$\begin{aligned}
|A \cap B| = |A| + |B| - |A \cup B| &\geqslant \binom{n-1}{t} + 2m - \lfloor m/2 \rfloor - \binom{n}{t} \\
&\geqslant m + m/2 - \binom{n-1}{t-1} \geqslant m + \frac{1}{2} \cdot \frac{\binom{n}{t} - 2}{2} - \binom{n-1}{t-1} \\
&= m + \left(\frac{n}{4t} - 1\right)\binom{n-1}{t-1} - \frac{1}{2} > m.
\end{aligned}$$

Then A contains at least one pair, i and $2i$, from B. □

Theorem 10. *Let $2 \leqslant t < n/4$ and $k_+ > k_- = 0$. Then $\mathcal{B}(n, t, k_+, 0)$ cannot lattice-tile $\mathbb{Z}^n$ when*

$$k_+ \geqslant 2\binom{n}{t} - 2.$$

Proof. By Theorem 2, suppose to the contrary that there is an Abelian group G with $|G| = \sum_{i=0}^{t}\binom{n}{i}k_+^i$ and a subset $S = \{s_1, s_2, \dots, s_n\} \subseteq G$ such that $G = M \diamond_t S$, where $M \triangleq [1, k_+]$. We consider the sums

$$x_1s_1 + x_2s_2 + \cdots + x_ts_t + x_{t+1}s_{t+1},$$

where $0 \leqslant x_1 < \binom{n}{t}$ and $0 \leqslant x_i \leqslant k_+$ for $i = 2, 3, \dots, t+1$. The total number of such sums is $\binom{n}{t}(k_+ + 1)^t$. Noting that

$$\binom{n}{t}(k_+ + 1)^t = \sum_{i=0}^{t}\binom{n}{t}\binom{t}{i}k_+^i > \sum_{i=0}^{t}\binom{n}{i}k_+^i = |G|,$$

there are two sums which are equal. Namely, there are two distinct vectors, $\mathbf{a} = (a_1, a_2, \dots, a_{t+1})$ and $\mathbf{a}' = (a'_1, a'_2, \dots, a'_{t+1})$, from $[0, \binom{n}{t} - 1] \times [0, k_+]^t$, such that

$$a_1s_1 + a_2s_2 + \cdots + a_{t+1}s_{t+1} = a'_1s_1 + a'_2s_2 + \cdots + a'_{t+1}s_{t+1}.$$

W.l.o.g., assume $a_1 \geqslant a'_1$. Let $b_i = a_i - a'_i$ for $i = 1, 2, \dots, t+1$. Rearranging the terms, we have

$$b_1s_1 + b_2s_2 + \cdots + b_{t+1}s_{t+1} = 0, \tag{3}$$

where $\mathbf{b} = (b_1, b_2, \dots, b_{t+1})$ is a non-zero vector from $[0, \binom{n}{t} - 1] \times [-k_+, k_+]^t$. Since $\binom{n}{t} - 1 \leqslant k_+$, to avoid contradicting the assumption $G = M \diamond_t S$, necessarily $b_i \geqslant 0$ for all $i = 2, 3, \dots, t+1$, i.e., $\mathbf{b} \in [0, k_+]^{t+1}$.

We now claim that there is a non-zero vector $\mathbf{v} \in [0, \binom{n}{t} - 1]^{t+1}$ such that $\mathbf{v} \cdot (s_1, s_2, \dots, s_{t+1}) = 0$. As a first step, we show that there is a non-zero vector $\mathbf{v} = (v_1, v_2, \dots, v_{t+1}) \in [0, k_+]^{t+1}$ such that $v_1, v_2 < \binom{n}{t}$ and $\mathbf{v} \cdot (s_1, s_2, \dots, s_{t+1}) = 0$. In (3), if $b_2 < \binom{n}{t}$, then $\mathbf{b}$ is the desired vector. Otherwise, $b_2 \geqslant \binom{n}{t}$. By symmetry (repeating the same arguments arriving in (3)), there is a non-zero vector $\mathbf{c} = (c_1, c_2, \dots, c_{t+1}) \in [0, k_+]^{t+1}$ with $c_2 < \binom{n}{t}$ such that $\mathbf{c} \cdot (s_1, s_2, \dots, s_{t+1}) = 0$. If $c_1 \geqslant \binom{n}{t}$, we consider the equation

$$(b_2 - c_2)s_2 + b_3s_3 + b_4s_4 + \cdots + b_{t+1}s_{t+1} = (c_1 - b_1)s_1 + c_3s_3 + c_4s_4 + \cdots + c_{t+1}s_{t+1},$$

which is obtained by rearranging $\mathbf{b} \cdot (s_1, s_2, \dots, s_{t+1}) = \mathbf{c} \cdot (s_1, s_2, \dots, s_{t+1})$. Note that $0 < c_1 - b_1 \leqslant k_+$, $0 < b_2 - c_2 \leqslant k_+$, and $b_i, c_i \in [0, k_+]$ for all $i = 3, 4, \dots, t+1$. This contradicts the assumption $G = M \diamond_t S$. Thus, necessarily, $c_1 < \binom{n}{t}$ and $\mathbf{c}$ is the desired vector. By using induction on the first j elements, $s_1, s_2, \dots, s_j$, we are able to show our claim.

Extending the arguments presented thus far, for any $2 \leqslant i_1 < i_2 < \cdots < i_t \leqslant n$, there is a non-zero vector

$$\mathbf{v}^{i_1,i_2,\dots,i_t} = (v_1^{i_1,i_2,\dots,i_t}, v_{i_1}^{i_1,i_2,\dots,i_t}, \dots, v_{i_t}^{i_1,i_2,\dots,i_t}) \in \left[0, \binom{n}{t} - 1\right]^{t+1}$$

such that

$$v_1^{i_1,i_2,\dots,i_t}s_1 + v_{i_1}^{i_1,i_2,\dots,i_t}s_{i_1} + \cdots + v_{i_t}^{i_1,i_2,\dots,i_t}s_{i_t} = 0.$$

Take any $2 \leqslant i'_1 < i'_2 < \cdots < i'_t \leqslant n$ such that $(i_1, i_2, \dots, i_t) \neq (i'_1, i'_2, \dots, i'_t)$. If there are two integers $v_1^{i_1,i_2,\dots,i_t}$ and $v_1^{i'_1,i'_2,\dots,i'_t}$ which are equal, then we have

$$v_{i_1}^{i_1,i_2,\dots,i_t}s_{i_1} + v_{i_2}^{i_1,i_2,\dots,i_t}s_{i_2} + \cdots + v_{i_t}^{i_1,i_2,\dots,i_t}s_{i_t} = v_{i'_1}^{i'_1,i'_2,\dots,i'_t}s_{i'_1} + v_{i'_2}^{i'_1,i'_2,\dots,i'_t}s_{i'_2} + \cdots + v_{i'_t}^{i'_1,i'_2,\dots,i'_t}s_{i'_t}.$$

To avoid contradicting the assumption that $G = M \diamond_t S$, necessarily, $v_{i_j}^{i_1,i_2,\dots,i_t} = v_{i'_j}^{i'_1,i'_2,\dots,i'_t} = 0$ for all $1 \leqslant j \leqslant t$, which in turn implies $v_1^{i_1,i_2,\dots,i_t} = 0$. This contradicts the fact that $\mathbf{v}^{i_1,i_2,\dots,i_t}$ is a non-zero vector. Therefore, the $\binom{n-1}{t}$ integers $v_1^{i_1,i_2,\dots,i_t}$ must be pairwise distinct.

Note that when $2 \leqslant t < n/4$, we have

$$\left(\frac{n}{4t} - 1\right)\binom{n-1}{t-1} > \frac{1}{2}.$$

By Lemma 1, there are $v_1^{i_1,i_2,\dots,i_t}$ and $v_1^{i'_1,i'_2,\dots,i'_t}$ such that $v_1^{i_1,i_2,\dots,i_t} = 2v_1^{i'_1,i'_2,\dots,i'_t} \neq 0$. Therefore,

$$\begin{aligned}
&v_{i_1}^{i_1,i_2,\dots,i_t} s_{i_1} + v_{i_2}^{i_1,i_2,\dots,i_t} s_{i_2} + \cdots + v_{i_t}^{i_1,i_2,\dots,i_t} s_{i_t} \\
&= 2v_{i'_1}^{i'_1,i'_2,\dots,i'_t} s_{i'_1} + 2v_{i'_2}^{i'_1,i'_2,\dots,i'_t} s_{i'_2} + \cdots + 2v_{i'_t}^{i'_1,i'_2,\dots,i'_t} s_{i'_t}.
\end{aligned}$$

Note that $\{i_1, i_2, \dots, i_t\} \neq \{i'_1, i'_2, \dots, i'_t\}$, $0 \leqslant v_{i_j}^{i_1,i_2,\dots,i_t} \leqslant \binom{n}{t} - 1 \leqslant k_+$ and $0 \leqslant 2v_{i'_j}^{i'_1,i'_2,\dots,i'_t} \leqslant 2\binom{n}{t} - 2 \leqslant k_+$. To avoid contradicting the assumption, necessarily $v_{i_j}^{i_1,i_2,\dots,i_t} = v_{i'_j}^{i'_1,i'_2,\dots,i'_t} = 0$ for all $1 \leqslant j \leqslant t$, and so $v_1^{i_1,i_2,\dots,i_t} = 0$. This contradicts the fact that $\mathbf{v}^{i_1,i_2,\dots,i_t}$ is a non-zero vector, which completes our proof. □

Unlike the other proofs in this section, the next one uses a geometric argument.

Theorem 11. *Let $\frac{2}{3}(n-1) \leqslant t \leqslant n-3$. Then $\mathcal{B}(n, t, k_+, 0)$ cannot lattice-tile $\mathbb{Z}^n$ when $k_+ \geqslant 2$.*

Proof. Suppose to the contrary that there is a lattice $\Lambda \subseteq \mathbb{Z}^n$ such that $\mathcal{B}$ tiles $\mathbb{Z}^n$ by Λ. According to the first two paragraphs in the proof of Theorem 6, we may assume that $\mathbf{1} \in \mathbf{a} + \mathcal{B}$, where

$$\mathbf{a} \triangleq (\underbrace{1, 1, \dots, 1}_{t+1}, a_{t+2}, \dots, a_n) \in \Lambda,$$

where $1 - k_+ \leqslant a_i \leqslant 1$ for $t+2 \leqslant i \leqslant n$.

Let $\tau \triangleq n - t$. The assumption $\frac{2}{3}(n-1) \leqslant t \leqslant n-3$ implies $\tau \geqslant 3$ and $2\tau - 2 \leqslant t$. We consider the vector

$$\mathbf{v} \triangleq (\underbrace{0, 0, \dots, 0}_{\tau-1}, \underbrace{1, 1, \dots, 1}_{t+1}).$$

Since $wt(\mathbf{v}) = t+1$ and $\tau - 1 \geqslant 1$, neither $\mathcal{B}$ nor $\mathbf{a} + \mathcal{B}$ contains $\mathbf{v}$. Thus there is another vector

$$\mathbf{b} = (b_1, b_2, \dots, b_n) \in \Lambda$$

such that $\mathbf{v} \in \mathbf{b} + \mathcal{B}$, where $-k_+ \leqslant b_i \leqslant 0$ for $1 \leqslant i \leqslant \tau - 1$ and $1 - k_+ \leqslant b_i \leqslant 1$ for $\tau \leqslant i \leqslant n$. In the following, we further narrow down the range of b_i.

1. $b_i = 1$ for all $\tau \leqslant i \leqslant n$. Otherwise, w.l.o.g., assume $b_\tau \leqslant 0$. Note that $\mathbf{v} \in \mathbf{b} + \mathcal{B}$. Then $(\underbrace{0, 0, \dots, 0}_{\tau}, \underbrace{1, 1, \dots, 1}_{t}) \in \mathbf{b} + \mathcal{B}$, contradicting $(\underbrace{0, 0, \dots, 0}_{\tau}, \underbrace{1, 1, \dots, 1}_{t}) \in \mathcal{B}$.
2. There is at least one $b_i = -k_+$ for some $1 \leqslant i \leqslant \tau - 1$. Otherwise, $-k_+ < b_i \leqslant 0$ for all $1 \leqslant i \leqslant \tau - 1$. Note that $\tau - 1 \leqslant t$ and we have shown $b_i = 1$ for all $\tau \leqslant i \leqslant n$. It follows that $\mathbf{1} \in \mathbf{b} + \mathcal{B}$, which contradicts $\mathbf{1} \in \mathbf{a} + \mathcal{B}$.

According to the argument above, by permuting the first $\tau - 1$ elements of $\mathbf{b}$, we may assume

$$\mathbf{b} = (-k_+, \underbrace{0, \dots, 0}_{p}, \underbrace{b_{p+2}, \dots, b_{\tau-1}}_{q}, \underbrace{1, 1, \dots, 1}_{t+1}),$$

where $p, q \geqslant 0$, $p + q = \tau - 2$ and $-k_+ \leqslant b_i \leqslant -1$ for $p+2 \leqslant i \leqslant \tau - 1$.

Now, for $0 \leqslant \ell \leqslant p$, define

$$\mathbf{u}_\ell \triangleq (1, \underbrace{0, 0, \dots, 0}_{p}, \underbrace{1, 1, \dots, 1}_{q+\ell}, \underbrace{0, 0, \dots, 0}_{q+1}, \underbrace{1, 1, \dots, 1}_{n-p-2q-\ell-2}).$$

There are $n - p - q - 1 = t + 1$ ones in $\mathbf{u}_\ell$ and so $\mathbf{u}_\ell$ is not contained in $\mathcal{B}$. Noting that $1 + p + q + \ell + q + 1 \leqslant 2 + 2(p+q) = 2\tau - 2 \leqslant t + 1$, there are $\tau - 1$ zeros in the first $t+1$ entries of $\mathbf{u}_\ell$, and so $\mathbf{u}_\ell \notin \mathbf{a} + \mathcal{B}$. The first entry of $\mathbf{u}_\ell$ is 1 while the first entry of $\mathbf{b}$ is $-k_+$. Thus, $\mathbf{u}_\ell \notin \mathbf{b} + \mathcal{B}$.

Assume $\mathbf{u}_\ell \in \mathbf{c}_\ell + \mathcal{B}$ for some $\mathbf{c}_\ell \in \Lambda$. According to the argument above, necessarily $\mathbf{c}_\ell \notin \{\mathbf{0}, \mathbf{a}, \mathbf{b}\}$. Since both $\mathbf{u}_\ell$ and $\mathbf{v}$ have $\tau - 1$ zeros in the first $t+1$ entries and ones in all the other entries and $\mathbf{a}$ has ones in the first $t+1$ entries, according to the symmetry, $\mathbf{c}_\ell$ has the same form as $\mathbf{b}$, namely,

$$\mathbf{c}_\ell = (1, \underbrace{*, *, \dots, *}_{p}, \underbrace{1, 1, \dots, 1}_{q+\ell}, \underbrace{*, *, \dots, *}_{q+1}, \underbrace{1, 1, \dots, 1}_{n-p-2q-\ell-2}),$$

where the entries marked with $*$ are in $[-k_+, 0]$ and at least one of them is $-k_+$.

We claim that all the last $q+1$ entries marked with $*$ in $\mathbf{c}_\ell$ should be 0. Otherwise, w.l.o.g., assume the first of them is negative, i.e.,

$$\mathbf{c}_\ell = (1, \underbrace{*, *, \dots, *}_{p}, \underbrace{1, 1, \dots, 1}_{q+\ell}, \underbrace{-x, *, \dots, *}_{q+1}, \underbrace{1, 1, \dots, 1}_{n-p-2q-\ell-2}),$$

where $1 \leqslant x \leqslant k_+$. Then

$$\mathbf{b} + \mathbf{c}_\ell = (1 - k_+, \underbrace{*, *, \dots, *}_{p}, \underbrace{b_{p+2} + 1, b_{p+3} + 1, \dots, b_{\tau-1} + 1}_{q}, \underbrace{2, 2, \dots, 2}_{\ell},$$
$$\underbrace{1 - x, \circledast, \dots, \circledast}_{q+1}, \underbrace{2, 2, \dots, 2}_{n-p-2q-\ell-2}),$$

where the entries marked with $*$ are in $[-k_+, 0]$ and the entries marked with $\circledast$ are in $[1 - k_+, 1]$. Note that $-k_+ \leqslant b_i \leqslant -1$ for $p + 2 \leqslant i \leqslant \tau - 1$, and $1 + p + q + q + 1 \leqslant 2(\tau - 2) + 2 = 2\tau - 2 \leqslant t$. It follows that

$$(\underbrace{0, 0, \dots, 0}_{1+p+q=\tau-1}, \underbrace{2, 2, \dots, 2}_{\ell}, 0, \underbrace{1, 1, \dots, 1}_{q}, \underbrace{2, 2, \dots, 2}_{n-p-2q-\ell-2}) \in \mathbf{b} + \mathbf{c}_\ell + \mathcal{B}.$$

Since $k_+ \geqslant 2$, the vector above is also contained in $\mathcal{B}$. Then we got $\mathbf{b} + \mathbf{c}_\ell = \mathbf{0}$, which contradicts that the first entry of $\mathbf{b} + \mathbf{c}_\ell$ is $1 - k_+ \leqslant -1$. Therefore,

$$\mathbf{c}_\ell = (1, \underbrace{*, *, \dots, *}_{p}, \underbrace{1, 1, \dots, 1}_{q+\ell}, \underbrace{0, 0, \dots, 0}_{q+1}, \underbrace{1, 1, \dots, 1}_{n-p-2q-\ell-2}).$$

Recall that the entries marked with $*$ are in $[-k_+, 0]$ and at least one of them is $-k_+$. Necessarily $p \geqslant 1$. Since there are $p+1$ choices of ℓ, at least two vectors, say $\mathbf{c}_{\ell_1}$ and $\mathbf{c}_{\ell_2}$, have $-k_+$ in the same entry. By permuting the p entries marked with $*$, assume both $\mathbf{c}_{\ell_1}$ and $\mathbf{c}_{\ell_2}$ have $-k_+$ in the first entry marked with $*$. Then

$$(1, \underbrace{-k_+, 0, \dots, 0}_{p}, \underbrace{1, 1, \dots, 1}_{n-p-1}) \in (\mathbf{c}_{\ell_1} + \mathcal{B}) \cap (\mathbf{c}_{\ell_2} + \mathcal{B}),$$

as $p - 1 + q + 1 = \tau - 2 \leqslant t$. It follows that $\mathbf{c}_{\ell_1} = \mathbf{c}_{\ell_2}$. W.l.o.g., assume $\ell_1 < \ell_2$. Then the $(n - p - 2q - \ell_2 - 1)$-th entry, from the right side, of $\mathbf{c}_{\ell_2}$ is 0, while the corresponding entry of $\mathbf{c}_{\ell_1}$ is 1, a contradiction. □

Continuing our specialization, we turn to tackle the case of $t = 2$, and present a strong restriction on the dimension n.

Theorem 12. *For any $k_+ \geqslant k_- \geqslant 0$, if $\mathcal{B}(n, 2, k_+, k_-)$ lattice-tiles $\mathbb{Z}^n$ and also $|\mathcal{B}(n, 2, k_+, k_-)|$ is even, then*

$$n = \frac{4\ell^2 - (k_+ + k_- - 3)^2 + 8}{4(k_+ + k_-)},$$

for some $\ell \in \mathbb{Z}$.

Proof. By Theorem 2 there exists an Abelian group G whose size is $|G| = |\mathcal{B}(n, 2, k_+, k_-)|$ such that $G = M \diamond_2 S$ for some $S \subseteq G$, $|S| = n$, where $M \triangleq [-k_-, k_+]^*$. Since G is Abelian and of even order, necessarily $G = \mathbb{Z}_{2^r} \times G'$, for some $r \geqslant 1$. We may therefore write any element $g \in G$ as a pair (a, b) where $a \in \mathbb{Z}_{2^\ell}$ and $b \in G'$, and we say g is *even* if $a \equiv 0 \pmod 2$, and *odd* otherwise.

Denote by n_1 the number of odd elements in S. Additionally, denote by $m_0 \triangleq \lfloor k_+/2 \rfloor + \lfloor k_-/2 \rfloor$ (respectively, $m_1 \triangleq \lceil k_+/2 \rceil + \lceil k_-/2 \rceil$) the number of even (respectively, odd) numbers in M.

Let us examine how the $\frac{1}{2}(\binom{n}{2}(k_+ + k_-)^2 + n(k_+ + k_-) + 1)$ odd elements of G are obtained via the 2-splitting. There are three possible ways:

1. An odd element in S times an odd number in M.
2. An odd element in S times an odd number in M, plus an even element in S times any number from M.
3. An odd element in S times an odd number in M, plus a different odd element in S times an even number from M.

Thus,

$$\begin{aligned}&n_1 m_1 + n_1 m_1 (n - n_1)(m_0 + m_1) + n_1 m_1 (n_1 - 1) m_0 \\ &= \frac{1}{2}\left(\binom{n}{2}(m_0 + m_1)^2 + n(m_0 + m_1) + 1\right).\end{aligned}$$

Solving for n_1 we obtain

$$n_1 = \frac{n(m_0 + m_1) - m_0 + 1 \pm \sqrt{n(m_1^2 - m_0^2) + m_0^2 - 2m_0 - 1}}{2m_1}. \tag{4}$$

We recall that $m_0 + m_1 = k_+ + k_-$. Additionally, we note that

$$|\mathcal{B}(n, 2, k_+, k_-)| = \binom{n}{2}(k_+ + k_-)^2 + n(k_+ + k_-) + 1$$

is even, which implies that $k_+ + k_-$ is odd, and then $m_1 - m_0 = 1$. It follows that $m_1^2 - m_0^2 = (m_1 - m_0)(m_1 + m_0) = m_1 + m_0 = k_+ + k_-$. Substituting back in (4), we use the fact that the square root must be an integer $\ell \in \mathbb{Z}$ to obtain the desired claim after some simple rearranging. □

Finally, we focus on the smallest case not studied before — tiling $\mathcal{B}(n, 2, 1, 0)$. In this case, by a careful study of the possible group splittings we obtain a full classification of possible tilings. We require some structural lemmas first. These hold for a weaker structure than a t-splitting: If in Definition 1 only the first condition holds, we denote it as $G \geqslant M \diamond_t S$.

Lemma 2. *Suppose that $G \geqslant \{1\} \diamond_2 S$. Let $n = |S|$. Consider the $(n + 1)n$ differences $s - s'$, where $s, s' \in S \cup \{0\}$ and $s \neq s'$. If there are two differences which are equal, then they must have the form*

$$s_i - s_j = s_k - s_i,$$

for some s_i, s_j and $s_k \in S \cup \{0\}$. Furthermore, if $s_i = 0$, then we must have $s_j = s_k$.

Proof. Assume that there are two distinct pairs $(s_i, s_j), (s_k, s_\ell) \in (S \cup \{0\})^2$ with $s_i \neq s_j$ and $s_k \neq s_\ell$ such that

$$s_i - s_j = s_k - s_\ell.$$

Rearranging the terms, we have

$$s_i + s_\ell = s_k + s_j.$$

Since $G \geqslant \{1\} \diamond_2 S$, $(s_i, s_j) \neq (s_k, s_\ell)$ and $\{s_i, s_\ell\} \neq \{s_k, s_j\}$, either $s_i = s_\ell$ or $s_k = s_j$. Then the conclusion follows. □

Lemma 3. *Suppose that $G \geqslant \{1\} \diamond_2 S$. For each $s_i \in S$, there is at most one unordered pair $\{s_j, s_k\} \subset S \cup \{0\}$ with $s_j \neq s_k$ such that*

$$s_i - s_j = s_k - s_i.$$

Proof. Suppose that there is another pair $\{s'_j, s'_k\}$ with $s'_j \neq s'_k$ such that $s_i - s'_j = s'_k - s_i$. Then

$$2s_i = s_j + s_k = s'_j + s'_k.$$

Since $s_j \neq s_k$, $s'_j \neq s'_k$ and $G \geqslant \{1\} \diamond_2 S$, necessarily $\{s_j, s_k\} = \{s'_j, s'_k\}$. □

For an Abelian group G, let $m_2(G)$ be the number of elements of order 2 in G, i.e.,

$$m_2(G) \triangleq |\{x \in G \; : \; x \neq 0, 2x = 0\}|.$$

Lemma 4. *Suppose that $G \geqslant \{1\} \diamond_2 S$, and let $n \triangleq |S|$. Then we have*

$$|G| + m_2(G) \geqslant n^2 - n + 1.$$

Proof. Denote

$$\Delta \triangleq \left\{(s, s') \; : \; s, s' \in S \cup \{0\} \text{ and } s \neq s'\right\}.$$

According to Lemma 3, for each $s_i \in S$, there is at most one unordered pair $\{s_j, s_k\} \subset S \cup \{0\}$ with $s_j \neq s_k$ such that $s_i - s_j = s_k - s_i$ (and so $s_i - s_k = s_j - s_i$). If such a pair exists, we remove (s_k, s_i) and (s_j, s_i) from Δ. Denote the remaining set as Δ'. Then $|\Delta'| \geqslant (n + 1)n - 2n$.

According to Lemma 2 and the definition of Δ', if there are two pairs in Δ' whose differences are equal, they must have the form $s_i - s_j = s_j - s_i$, and so, $2(s_i - s_j) = 0$. Hence, for every $g \in G$ of order 2, there are at most two pairs $(s, s') \in \Delta'$ with $s - s' = g$ and, for every other non-zero element of G, there is at most one such representation. It follows that

$$|G| - 1 + m_2(G) \geqslant (n + 1)n - 2n.$$

Rearranging the terms, we complete the proof. □

Theorem 13. *Let $n \geqslant 3$. Then $\mathcal{B}(n, 2, 1, 0)$ lattice-tiles $\mathbb{Z}^n$ only when $n \in \{3, 5\}$, and only by 2-splitting $\mathbb{Z}_7$ and 2-splitting $\mathbb{F}_2^4$, respectively.*

Proof. By Lemma 4, if we are to have a splitting $G = \{1\} \diamond_2 S$, then

$$\binom{n}{2} + n + 1 + m_2(G) \geqslant n^2 - n + 1,$$

where we used the fact that $G = \{1\} \diamond_2 S$ implies $|G| = |\mathcal{B}(n, 2, 1, 0)|$. Rearranging we get,

$$m_2(G) \geqslant \frac{1}{2}n(n - 3). \tag{5}$$

We now turn to look at G. Since it is Abelian, we may write

$$G = \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_\ell},$$

with $n_1, \ldots, n_\ell \geqslant 2$. We observe that

$$m_2(\mathbb{Z}_{n_i}) = \begin{cases} 0 & n_i \text{ is odd,} \\ 1 & \text{otherwise.} \end{cases}$$

Thus,

$$m_2(G) = \left(\prod_{i=1}^{\ell}(m_2(\mathbb{Z}_{n_i}) + 1)\right) - 1.$$

If $G \neq \mathbb{F}_2^r$, then necessarily

$$m_2(G) \leqslant \frac{1}{2}|G| - 1 = \frac{1}{4}(n^2 + n - 2), \tag{6}$$

which is attained by setting exactly one of the n_i to be 4, and the rest to be 2.

If we compare (5) and (6), then for $n \geqslant 7$ the lower bound of (5) is greater than the upper bound of (6), hence, only $G = \mathbb{F}_2^r$ is still possible. For $n \leqslant 6$ we deal with the cases separately:

- For $n = 3$, $|G| = 7$, hence $G = \mathbb{Z}_7$. A splitting set $S = \{1, 2, 4\}$ can be found in [1, Theorem 6].
- For $n = 4$, $|G| = 11$, hence $G = \mathbb{F}_{11}$, but $m_2(G) = 0$, contradicting (5).
- For $n = 5$, $|G| = 16$, with the following options:
  - $G = \mathbb{F}_2^4$, for which $m_2(G) = 15$, and a 2-splitting exists by Theorem 3 (see Example 2).
  - $G = \mathbb{Z}_4 \times \mathbb{F}_2^2$, for which $m_2(G) = 7$, but a computer search rules out such a splitting.
  - $G = \mathbb{Z}_4^2$, for which $m_2(G) = 3$, contradicting (5).
  - $G = \mathbb{Z}_8 \times \mathbb{F}_2$, for which $m_2(G) = 3$, contradicting (5).
  - $G = \mathbb{Z}_{16}$, for which $m_2(G) = 1$, contradicting (5).
- For $n = 6$, $|G| = 22$, hence $G = \mathbb{F}_2 \times \mathbb{F}_{11}$, but $m_2(G) = 1$, contradicting (5).

Finally, if $n \geqslant 7$, only $G = \mathbb{F}_2^r$ remains an option, but by Theorem 4 we must then have a perfect $[n, k, 5]$ linear code over $\mathbb{F}_2$, and such codes do not exist (e.g., see [6]). $\square$

Using a similar method, we now direct our attention to the case of $\mathcal{B}(n, 2, 2, 0)$. Let G be an Abelian group and assume that $G \geqslant \{1, 2\} \diamond_2 S$, for some $S = \{s_1, s_2, \dots, s_n\} \subseteq G$. Denote $s_{n+i} \triangleq 2s_i$ for $1 \leqslant i \leqslant n$ and $s_\infty \triangleq 0$. Consider the congruence modulo n. We assume that $\infty \equiv \infty \pmod{n}$, and $\infty \not\equiv i \pmod{n}$ and $i \not\equiv \infty \pmod{n}$ for all $i \in [1, 2n]$. Let

$$\Delta \triangleq \left\{(s_i, s_j) \ : \ i, j \in [1, 2n] \cup \{\infty\}, i \not\equiv j \ (\mathrm{mod}\ n)\right\}.$$

Then $|\Delta| = (2n + 1)2n - 2n = 4n^2$. We are to estimate the number of the equations

$$s_i - s_j = s_k - s_\ell,$$

where $(s_i, s_j), (s_k, s_\ell) \in \Delta$ and $(i, j) \neq (k, \ell)$. Note that the equation implies

$$s_i + s_\ell = s_k + s_j.$$

Since $G \geqslant \{1, 2\} \diamond_2 S$, either $i \equiv \ell \pmod{n}$ or $k \equiv j \pmod{n}$. By exchanging the two sides of the equations, we assume that $i \equiv \ell \pmod{n}$ always holds.

Lemma 5. *In the setting above, the number of the equations*

$$s_i - s_j = s_k - s_\ell,$$

where $(s_i, s_j), (s_k, s_\ell) \in \Delta$, $i \equiv \ell \pmod{n}$ and $k \not\equiv j \pmod{n}$, is at most $8n$.

Proof. If $i = \ell = \infty$, then $s_j + s_k = 0$. Since $G \geqslant \{1, 2\} \diamond_2 S$, necessarily $j \equiv k \pmod{n}$, contradicting the assumption.

Now, let $\bar{i}$ be the unique integer of $[1, n]$ such that $\bar{i} \equiv i \equiv \ell \pmod{n}$.

1. If $i = \ell = \bar{i}$, then $2s_{\bar{i}} = s_k + s_j$. Since $G \geqslant \{1, 2\} \diamond_2 S$, necessarily $(k, j) \in \left\{(\bar{i} + n, \infty), (\infty, \bar{i} + n)\right\}$.
2. If $i = \bar{i}$ and $\ell = n + \bar{i}$, then $s_{\bar{i}} - s_j = s_k - 2s_{\bar{i}}$. We claim that there is at most one pair $\{j, k\}$ with $j \not\equiv k \pmod{n}$ such that the equality holds; otherwise, suppose we have another

pair $\{j', k'\}$ satisfying the conditions, then $s_j + s_k = s_{j'} + s_{k'}$, contradicting the fact that $G \geqslant \{1, 2\} \diamond_2 S$.
3. If $(i, \ell) = (\bar{i} + n, \bar{i})$ or $(\bar{i} + n, \bar{i} + n)$, we have the same claim as that in case 2.

According to the argument above, given $\bar{i} \in [1, n]$, if $i \equiv \ell \equiv \bar{i} \pmod n$, we have at most four pairs $\{j, k\}$ such that the equation holds. The conclusion follows since each pair can generate two equations. □

Let $m_3(G)$ be the number of elements of order 3 in G, i.e.,

$$m_3(G) \triangleq |\{x \in G \ : \ x \neq 0, 3x = 0\}|.$$

Lemma 6. *In the setting above, further assume that the order of G is odd. Then the number of the equations*

$$s_i - s_j = s_k - s_\ell,$$

where $(s_i, s_j), (s_k, s_\ell) \in \Delta$, $i \equiv \ell \pmod n$ *and* $k \equiv j \pmod n$, *is at most*

$$2m_3(G) + 11n + 11.$$

Proof. Let $\bar{i}, \bar{j} \in [1, n] \cup \{\infty\}$ such that $\bar{i} \equiv i \equiv \ell \pmod n$ and $\bar{j} \equiv j \equiv k \pmod n$. By the definition of Δ, we have $i \not\equiv j \pmod n$, and so, $\bar{i} \neq \bar{j}$. The equation $s_i - s_j = s_k - s_\ell$ implies that

$$as_{\bar{i}} - bs_{\bar{j}} = cs_{\bar{j}} - ds_{\bar{i}},$$

for some $a, b, c, d \in \{1, 2\}$. We discuss the number of equations for each possible value of (a, b, c, d).

1. If $a = b = c = d = 1$, then $2s_{\bar{i}} = 2s_{\bar{j}}$, contradicting $G \geqslant \{1, 2\} \diamond_2 S$.
2. If $a + d = 2$, then there are at most $n + 1$ ordered pairs $(\bar{i}, \bar{j})$ such that the equation holds; otherwise, by the pigeonhole principle there exist two ordered pairs $(\bar{i}, \bar{j})$, and $(\bar{i}', \bar{j})$ satisfying the equation, with $\bar{i} \neq \bar{i}'$. Then we get that $(a + d)s_{\bar{i}} = (b + c)s_{\bar{j}} = (a + d)s_{\bar{i}'}$, i.e., $2s_{\bar{i}} = 2s_{\bar{i}'}$ for some $\bar{i} \neq \bar{i}'$, a contradiction.
3. If $a + d = 4$, then again there are at most $n + 1$ ordered pairs $(\bar{i}, \bar{j})$ such that the equation holds; otherwise, we have $4s_{\bar{i}} = 4s_{\bar{i}'}$ for some $\bar{i} \neq \bar{i}'$, contradicting the assumption that $|G|$ is odd.
4. If $b + c = 2$ or 4, we have the same claim as that in cases 2 and 3.
5. If $(a, b, c, d) = (2, 2, 1, 1)$ or $(1, 1, 2, 2)$, then

$$2s_{\bar{i}} - 2s_{\bar{j}} = s_{\bar{j}} - s_{\bar{i}}$$

and

$$s_{\bar{i}} - s_{\bar{j}} = 2s_{\bar{j}} - 2s_{\bar{i}}.$$

Rearranging the terms, we have $3(s_{\bar{j}} - s_{\bar{i}}) = 0$ and $3(s_{\bar{i}} - s_{\bar{j}}) = 0$. Thus the total number of such two kinds of equations is at most $m_3(G)$.
6. If $(a, b, c, d) = (2, 1, 2, 1)$ or $(1, 2, 1, 2)$, then

$$2s_{\bar{i}} - s_{\bar{j}} = 2s_{\bar{j}} - s_{\bar{i}}$$

and

$$s_{\bar{i}} - 2s_{\bar{j}} = s_{\bar{j}} - 2s_{\bar{i}}.$$

If the equations above occur, then the equations in case 5 also occur. Thus the total number of such two kinds of equations is also at most $m_3(G)$.

Note that cases 2,3 and 4 include $2^4 - 4 - 1 = 11$ possible values of (a, b, c, d). The conclusion follows by summing up all the numbers discussed above. □

Lemma 7. *Suppose that $G \geqslant \{1,2\} \diamond_2 S$, and let $n \triangleq |S|$. If $|G|$ is odd, then we have that*

$$|G| + 2m_3(G) \geqslant 4n^2 - 19n - 10.$$

Proof. Combining Lemmas 5 and 6, we repeat the same arguments as in Lemma 4 to obtain the result. □

We can now state and prove the result on $\mathcal{B}(n, 2, 2, 0)$.

Theorem 14. *Let $n \geqslant 3$, then $\mathcal{B}(n, 2, 2, 0)$ lattice-tiles $\mathbb{Z}^n$ only when $n \in \{3, 11\}$, and only by 2-splitting $\mathbb{Z}_{19}$ and 2-splitting $\mathbb{F}_3^5$, respectively.*

Proof. Note that $|\mathcal{B}(n, 2, 2, 0)| = 2n^2 + 1$, which is odd. By Lemma 7, if we are to have a splitting $G = \{1, 2\} \diamond_2 S$, then

$$2n^2 + 1 + 2m_3(G) \geqslant 4n^2 - 19n - 10.$$

Rearranging we get,

$$m_3(G) \geqslant \frac{1}{2}(2n^2 - 19n - 11). \tag{7}$$

We now turn to look at G. Write

$$G = \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_\ell},$$

where $n_1, \dots, n_\ell \geqslant 3$, and all of them are odd. Then

$$m_3(G) = \left(\prod_{i=1}^{\ell}(m_3(\mathbb{Z}_{n_i}) + 1)\right) - 1.$$

Since

$$m_3(\mathbb{Z}_{n_i}) = \begin{cases} 2 & n_i \text{ is divisible by } 3, \\ 0 & \text{otherwise,} \end{cases}$$

if $G \neq \mathbb{F}_3^r$, then necessarily

$$m_3(G) \leqslant \frac{1}{3}|G| - 1 = \frac{2n^2 - 2}{3}, \tag{8}$$

which is attained by setting exactly one of the n_i to be 9, and the rest to be 3.

If we compare (7) and (8), then for $n \geqslant 30$ the lower bound of (7) is greater than the upper bound of (8), hence, only $G = \mathbb{F}_3^r$ is still possible. However, if $G = \mathbb{F}_3^r$, by Theorem 4 we must then have a perfect $[n, k, 5]$ linear code over $\mathbb{F}_3$, and such codes do not exist if $n \neq 11$ (e.g., see [6]).

For $11 \leqslant n \leqslant 29$, (7) implies $m_3(G) \geqslant \frac{1}{2}(n(2n - 19) - 11) \geqslant 11$. Necessarily 27 divides $|G|$. The only two possible cases are $n = 11$ with $|G| = 243$, and $n = 16$ with $|G| = 513$.

We deal with the remaining cases separately:

- For $n = 3$, $|G| = 19$, hence $G = \mathbb{Z}_{19}$. A splitting set $S = \{1, 11, 7\}$ can be found in [1, Theorem 6].
- For $n = 4$, the non-existence is shown in [1, Corollary 6].
- For $n \in \{5, 6, 8, 9, 10\}$, $|G|$ is square-free, hence G is cyclic. A computer search rules out these cases.
- For $n = 7$, $|G| = 99$, hence $G = \mathbb{Z}_9 \times \mathbb{Z}_{11}$ (which is isomorphic to $\mathbb{Z}_{99}$) or $\mathbb{F}_3 \times \mathbb{Z}_{33}$. A computer search rules out these two cases.
- For $n = 11$, $|G| = 243$, with the following options.
  - $G = \mathbb{F}_3^5$, for which $m_3(G) = 242$.
  - $G = \mathbb{Z}_9 \times \mathbb{F}_3^3$, for which $m_3(G) = 80$.

- $G = \mathbb{Z}_9 \times \mathbb{Z}_9 \times \mathbb{F}_3$, for which $m_3(G) = 26$.
- $G = \mathbb{Z}_{27} \times \mathbb{F}_3^2$, for which $m_3(G) = 26$.
- $G = \mathbb{Z}_{27} \times \mathbb{Z}_9$, for which $m_3(G) = 8$, contradicting (7).
- $G = \mathbb{Z}_{81} \times \mathbb{F}_3$, for which $m_3(G) = 8$, contradicting (7).
- $G = \mathbb{Z}_{243}$, for which $m_3(G) = 2$, contradicting (7).

A computer search rules out the groups $\mathbb{Z}_9 \times \mathbb{F}_3^3$, $\mathbb{Z}_9 \times \mathbb{Z}_9 \times \mathbb{F}_3$ and $\mathbb{Z}_{27} \times \mathbb{F}_3^2$. When $G = \mathbb{F}_3^5$, a 2-splitting exists by Theorem 3 (see Example 4).

- For $n = 16$, $|G| = 513 = 27 \times 19$, hence $m_3(G) \leqslant 26$, contradicting (7). $\square$

5. Conclusion

In this paper we studied general tilings as well as lattice tilings of $\mathbb{Z}^n$ with $\mathcal{B}(n, t, k_+, k_-)$. These may act as perfect error-correcting codes over a channel with at most t limited-magnitude errors. We constructed such lattice tilings from perfect codes in the Hamming metric, and provided several non-existence results. We summarize some of our non-existence results for lattice tilings and below, where it is interesting to note the difference between the cases of $\frac{t}{n} < \frac{1}{2}$ and $\frac{t}{n} \geqslant \frac{1}{2}$.

Corollary 2. *Let $2 \leqslant t < n/2$, and $k_+ \geqslant k_- \geqslant 0$ not both 0. Then $\mathcal{B}(n, t, k_+, k_-)$ cannot lattice-tile $\mathbb{Z}^n$ when one of the following holds:*

1. $\frac{(k_-+1)^2}{k_++k_-+1} \geqslant \binom{n}{t}^{1/t}$.
2. *$t < n/4$, $k_- = 0$ and $k_+ \geqslant 2\binom{n}{t} - 2$.*
3. *$t = 2$, $k_- = 0$, $k_+ = 1$ and $n \neq 5$.*
4. *$t = 2$, $k_- = 0$, $k_+ = 2$ and $n \neq 11$.*

Corollary 3. *Let $2 \leqslant t < n \leqslant 2t$, and $k_+ \geqslant k_- \geqslant 0$ not both 0. If $\mathcal{B}(n, t, k_+, k_-)$ lattice-tiles $\mathbb{Z}^n$, then one of the following holds:*

1. *$k_- = 0$ and one of the following holds:*
    1. *$t = n - 1$(such tilings have been constructed in [1,11]);*
    2. *$(2n - 2)/3 \leqslant t \leqslant n - 3$ and $k_+ = 1$.*[2]*;*
    3. *$n/2 \leqslant t < (2n - 2)/3$;*
2. *$k_+ = k_-$ and one of the following holds:*
    1. *$(4n - 2)/5 \leqslant t \leqslant n - 1$ and $k_+ = k_- = 1$;*
    2. *$n/2 \leqslant t < (4n - 2)/5$ and $\sum_{i=1}^{t} \binom{n}{i}(2k_+)^{i-1} \geqslant (k_+ + 1)^t$.*

It is also interesting to compare the results here, when $t \geqslant 2$, with the known results for $t = 1$. The non-existence results we have here rely heavily on geometric arguments, or general algebraic arguments. The notable exceptions are Theorems 13 and 14, which carefully study the structure of the group being split. This is in contrast with the strong non-existence results when $t = 1$, due to the fact that when $t = 1$, if G is split then so is the cyclic group of the same size, $\mathbb{Z}_{|G|}$. This does not hold when $t \geqslant 2$, as evident, for example, during the proof of Theorem 13, where $\mathbb{F}_2^4$ is 2-split but $\mathbb{Z}_{16}$ is not.

Whether some strong statement may be said about the structure of the group being split, remains as an open question for further research. It is also interesting to ask whether more t-splittings exist, namely, whether t-splittings exist which are not derived from perfect codes in the Hamming metric. Finally, it remains open whether any other non-lattice tilings of $\mathcal{B}(n, t, k_+, k_-)$ exist.

[2] Recall that the entire case of $t = n - 2$ has been excluded in [1].

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