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Some Problems Concerning the Test Functions in the Szegö and Avram-Parter Theorems

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Abstract. The Szegö and Avram-Parter theorems give the limit of the arithmetic mean of the values of certain test functions at the eigenvalues and singular values of Toeplitz matrices as the matrix dimension increases to infinity. This paper is concerned with some questions that arise when the test functions do not satisfy the known growth restrictions at infinity or when the test function has a logarithmic singularity within the range of the symbol. Several open problems are listed and accompanied by a few new results that illustrate the delicacy of the matter.

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1. Introduction

Given an $n \times n$ matrix A, we denote by $s_1(A) \le \cdots \le s_n(A)$ the singular values of A, and if A is Hermitian, $A = A^*$, we let $\lambda_1(A) \le \cdots \le \lambda_n(A)$ stand for the eigenvalues of A. The $n \times n$ Toeplitz matrix generated by a function $a \in L^1(-\pi, \pi)$ is the matrix $T_n(a) := (a_{j-k})_{j,k=1}^n$ where a_j is the jth Fourier coefficient of a,

$$a_j = \int_{-\pi}^{\pi} a(\theta) e^{-ij\theta} \frac{d\theta}{2\pi} \quad (j \in \mathbb{Z}).$$

If a is real-valued, then $T_n(a)$ is Hermitian and the (first) Szegö limit theorem states that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} F(\lambda_j(T_n(a))) = \int_{-\pi}^{\pi} F(a(\theta)) \frac{d\theta}{2\pi} \tag{1}$$

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for appropriate functions $F : \mathbb{R} \to \mathbb{R}$. Theorems of the Avram-Parter type concern complex-valued functions a and say that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} F(s_j(T_n(a))) = \int_{-\pi}^{\pi} F(|a(\theta)|) \frac{d\theta}{2\pi} \tag{2}$$

for certain functions $F : [0,\infty) \to \mathbb{R}$. The functions F in (1) and (2) are referred to as test functions, and the problem consists in proving whether (1) and (2) or a modification of (1) and (2) is true for a given test function F. This paper is devoted to a few open questions pertaining to this problem, and it also contains some new results.

Here are two concrete problems we have been unable to solve. They provide the reader with an idea of the kind of questions considered in this paper.

Problem 1.1. *Let a be a real-valued trigonometric polynomial that assumes both positive and negative values and let $F(\lambda) = \log|\lambda|$ for $\lambda \neq 0$ and $F(0) = 0$. Is there a sequence $n_1 < n_2 < \cdots$ such that*

$$\lim_{k\to\infty} \frac{1}{n_k} \sum_{j=1}^{n_k} F(\lambda_j(T_{n_k}(a))) = \int_{-\pi}^{\pi} F(a(\theta)) \frac{d\theta}{2\pi} \ ?$$

Problem 1.2. *Let $F(s) = 0$ for $s \in [0,1]$ and $F(s) = s\log s$ for $s \in [1,\infty)$. Is (2) true for all $a \in L^1(-\pi,\pi)$ for which the right-hand side of (2) is finite?*

2. Extensions of the Avram-Parter theorem

We abbreviate $L^p(-\pi,\pi)$ to L^p and denote the function $\theta \mapsto F(|a(\theta)|)$ by $F(|a|)$. For simplicity, we assume in this section that $F \geq 0$. The question we are interested in is whether if $a \in L^1$ and $F : [0,\infty) \to [0,\infty)$ is continuous, does it follow that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} F(s_j(T_n(a))) = \begin{cases} \int_{-\pi}^{\pi} F(|a(\theta)|) \frac{d\theta}{2\pi} & \text{if} \quad F(|a|) \in L^1, \\ \infty & \text{if} \quad F(|a|) \notin L^1. \end{cases}$$

Clearly, this amounts to asking whether

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} F(s_j(T_n(a))) = \|F(|a|)\|_1, \tag{3}$$

where $\|\cdot\|_p$ is the norm in L^p, that is, $\|f\|_p^p = \int_{-\pi}^{\pi} |f(\theta)|^p \frac{d\theta}{2\pi}$. It is known that (3) holds if

(a) $a \in L^\infty$ and F is continuous (Parter [8], Avram [1]),
(b) $a \in L^1$ and F is bounded and uniformly continuous (Zamarashkin and Tyrtyshnikov [13], Tilli [12]),
(c) $a \in L^p$ $(1 \leq p < \infty)$ and F is continuous with $F(s) = O(s^p)$ as $s \to \infty$ (Serra Capizzano [10]).

Result (c) implies in particular that (3) is true for all $a \in L^1$ and all continuous $F : [0,\infty) \to [0,\infty)$ satisfying $F(s) = O(s)$. Thus, (b) is contained in (c).

Note also that in all these case $\|F(|a|)\|_1 < \infty$. The following result shows that (3) is always true if $\|F(|a|)\|_1 = \infty$.

Proposition 2.1. *Let $F : [0, \infty) \to [0, \infty)$ be a continuous function and let $a \in L^1$. If*

$$C := \liminf_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} F(s_j(T_n(a))) < \infty, \tag{4}$$

then $F(|a|) \in L^1$ and $\|F(|a|)\|_1 \le C$.

Proof. Fix $\varepsilon > 0$ and choose $n_1 < n_2 < \cdots$ so that

$$\frac{1}{n_k} \sum_{j=1}^{n_k} F(s_j(T_{n_k}(a))) < C + \varepsilon.$$

For a natural number M, define $F_M : [0, \infty) \to [0, \infty)$ by

$$F_M(s) = \begin{cases} F(s) & \text{for} \quad s \in [0, M], \\ (M + 1 - s)F(s) & \text{for} \quad s \in [M, M + 1], \\ 0 & \text{for} \quad s \in [M + 1, \infty). \end{cases}$$

The function F_M is continuous and has compact support. From (b) we therefore deduce that

$$\begin{aligned} \int_{-\pi}^{\pi} F_M(|a(\theta)|)\, d\theta &= \lim_{k\to\infty} \frac{1}{n_k} \sum_{j=1}^{n_k} F_M(s_j(T_{n_k}(a))) \\ &\le \limsup_{k\to\infty} \frac{1}{n_k} \sum_{j=1}^{n_k} F_M(s_j(T_{n_k}(a))) \le C + \varepsilon. \end{aligned} \tag{5}$$

Furthermore,

$$F_1(|a(\theta)|) \le F_2(|a(\theta)|) \le \cdots . \tag{6}$$

By the Beppo Levi theorem, (5) and (6) imply that

$$F(|a(\theta)|) = \lim_{M\to\infty} F_M(|a(\theta)|)$$

is a function in L^1 and that $\|F(|a|)\|_1 \le C + \varepsilon$. □

Corollary 2.2. *If $a \in L^1$, then (3) is true whenever $F : [0, \infty) \to [0, \infty)$ is continuous and $F(|a|) \notin L^1$.*

Proof. This is immediate from Proposition 2.1. □

Thus, if (3) would be valid whenever $F : [0, \infty) \to [0, \infty)$ is continuous and $F(|a|) \in L^1$, we could say that the Avram-Parter theorem is true whenever it makes sense. Unfortunately, this is not the case.

Proposition 2.3. *There exist continuous functions $F : [0, \infty) \to [0, \infty)$ and functions $a \in L^1$ such that $F(|a|) \in L^1$ but (3) is false.*

Proof. Let $a(\theta) = \theta^{-\alpha}$ for $\theta \in (-\pi, \pi)$. If $0 < \alpha < 1$, then $a \in L^1$. The maximal singular value of the positive definite Hermitian matrix $T_n(a)$ is $\lambda_n = \|T_n(a)\|$. Since $\|T_n(a)\|$ increases monotonically to $\|a\|_\infty = \infty$, there are $n_1 < n_2 < \cdots$ such that $0 < \lambda_{n_1} < \lambda_{n_2} < \cdots$ and $\lambda_{n_k} \to \infty$. Choose $\varepsilon_{n_k} > 0$ small enough and let $F : [0, \infty) \to [0, \infty)$ be the function which increases linearly from 0 to n_k^2 on $[\lambda_{n_k} - \varepsilon_{n_k}, \lambda_{n_k}]$, decreases linearly from n_k^2 to 0 on $[\lambda_{n_k}, \lambda_{n_k} + \varepsilon_{n_k}]$, and is identically zero outside $\cup_{k \geq 1}[\lambda_{n_k} - \varepsilon_{n_k}, \lambda_{n_k} + \varepsilon_{n_k}]$. We then have

$$\frac{1}{n_k} \sum_{j=1}^{n_k} F(s_j(T_{n_k}(a))) \geq \frac{1}{n_k} F(s_{n_k}(T_{n_k}(a))) = \frac{1}{n_k} F(\lambda_{n_k}) = n_k$$

and hence

$$\limsup_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} F(s_j(T_n(a))) = \infty.$$

On the other hand,

$$\begin{aligned} \|F(|a|)\|_1 &= 2 \int_0^\pi F(\theta^{-\alpha}) \frac{d\theta}{2\pi} = \frac{2}{\alpha} \int_{\pi^{-\alpha}}^\infty F(s) s^{-1/\alpha - 1} ds \\ &\leq \frac{2}{\alpha} \sum n_k^2 (\lambda_{n_k} - \varepsilon_{n_k})^{-1/\alpha - 1} (2\varepsilon_{n_k}) < \infty \end{aligned}$$

if only ε_{n_k} is chosen small enough. Consequently, the right-hand side of (3) is finite, but the limit on the left of (3) does not exist or is infinite. □

The function F constructed in the proof of Proposition 2.3 is not monotonous. This leads to the following question.

Problem 2.4. *Is* (3) *true for every monotonically increasing and continuous function* $F : [0, \infty) \to [0, \infty)$*?*

Corollary 2.2 in conjunction with (c) shows that the answer is in the affirmative if $F(s) \simeq s^p$ $(1 \leq p < \infty)$ as $s \to \infty$, which means that there are constants $0 < C_1 < C_2 < \infty$ such that $C_1 s^p \leq F(s) \leq C_2 s^p$ for all sufficiently large s. Here are some more test functions for which the answer is positive.

Proposition 2.5. *Let* μ *be a nonnegative Borel measure on* $[1, \infty)$ *such that*

$$F(s) := \int_1^\infty s^x \, d\mu(x) < \infty$$

for all $s \geq 0$*. Then* $F : [0, \infty) \to [0, \infty)$ *is a continuous, convex, and monotonically increasing function and* (3) *is true for this* F *and all* $a \in L^1$*.*

Proof. It is clear that F is nonnegative, monotonically increasing, and convex. This implies that F is continuous. By virtue of Corollary 2.2, it remains to prove (3) under the assumption that

$$\|F(|a|)\|_1 = \int_{-\pi}^{\pi} \int_1^\infty |a(\theta)|^x \, d\mu(x) \, \frac{d\theta}{2\pi} < \infty. \tag{7}$$

Since the iterated integral in (7) is finite, we deduce from Tonelli's theorem that $\int_{-\pi}^{\pi} |a(\theta)|^x \, d\theta < \infty$ for μ-almost all x in the support of the measure μ and that

$$\|F(|a|)\|_1 = \int_1^\infty \int_{-\pi}^{\pi} |a(\theta)|^x \frac{d\theta}{2\pi} \, d\mu(x). \tag{8}$$

It follows in particular that $a \in L^x$ for all $x \in \operatorname{supp} \mu$. Avram [1] proved that

$$\frac{1}{n} \sum_{j=1}^{n} s_j(T_n(a))^x \le \|a\|_x^x \tag{9}$$

for all $x \ge 1$. (This nice inequality was rediscovered and proved by different methods in [11].) Using (8) and (9) we get

$$\begin{aligned} \frac{1}{n} \sum_{j=1}^{n} F(s_j(T_n(a))) &= \int_1^\infty \frac{1}{n} \sum_{j=1}^{n} s_j(T_n(a))^x \, d\mu(x) \le \int_1^\infty \|a\|_x^x \, d\mu(x) \\ &= \int_1^\infty \int_{-\pi}^{\pi} |a(\theta)|^x \frac{d\theta}{2\pi} \, d\mu(x) = \|F(|a|)\|_1 \end{aligned}$$

and hence

$$\limsup_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} F(s_j(T_n(a))) \le \|F(|a|)\|_1.$$

Combining this estimate with Proposition 2.1 we arrive at (3). □

Corollary 2.6. *Let I be a finite subset of $[0,1]$ and J be a countable subset of $(1,\infty)$. For $p \in I \cup J$, let F_p be a positive real number. Suppose the series*

$$F(s) := \sum_{p \in I \cup J} F_p s^p$$

converges for every $s \in [0,\infty)$. Then $F : [0,\infty) \to [0,\infty)$ is a continuous and monotonically increasing function and (3) holds for this F and all $a \in L^1$.

Proof. Let $d\mu(x) = \sum_{p \in J} F_p \delta(x-p) \, dx$. Then

$$F(s) = \sum_{p \in I} F_p s^p + \int_1^\infty s^x \, d\mu(x) =: F_1(s) + F_2(s).$$

It is obvious that F_1 is nonnegative, continuous, and monotonically increasing. For F_2, these properties can be deduced from Proposition 2.5. Since $F_1(s) = O(s)$ as $s \to \infty$, result (c) implies that (3) is true for F_1 and all $a \in L^1$. Proposition 2.5 yields (3) for F_2 and all $a \in L^1$. □

Corollary 2.6 shows in particular that (3) is valid for all $a \in L^1$ if

$$F(s) = \exp(\alpha s^\beta) = \sum_{p=0}^{\infty} \frac{\alpha^p}{p!} s^{p\beta}$$

with α and β in $(0,\infty)$. Using Proposition 2.5 with $d\mu(x) = \chi_{(1,\alpha)}(x)\,dx$ $(\alpha > 1)$ we get (3) for all $a \in L^1$ and

$$F(s) = \int_1^\alpha s^x\,dx = \frac{s^\alpha - s}{\log s}.$$

Because $s/\log s = O(s)$ as $s \to \infty$, it follows from (c) that (3) holds for all $a \in L^1$ and $F(s) \simeq \frac{s^\alpha}{\log s}$ $(s \to \infty)$ provided $\alpha \in (1,\infty)$. Analogously (we omit the computational details), the choice $d\mu(x) = \chi_{(1,\alpha)}(x)(\alpha - x)^{\beta-1}\,dx$ $(\alpha > 1, \beta > 0)$ delivers (3) for all $a \in L^1$ and

$$F(s) \simeq \frac{s^\alpha}{(\log s)^\beta} \quad (s \to \infty) \tag{10}$$

with $\alpha \in (1,\infty)$ and $\beta \in (0,\infty)$. Taking $d\mu(x) = \chi_{(1,\alpha)}(x)(\alpha - x)^{\beta-1}|\log(\alpha - x)|^\gamma$ with $\alpha > 1$, $\beta > 0$, $\gamma \in (-\infty,\infty)$, we obtain

$$F(s) \simeq \frac{s^\alpha}{(\log s)^\beta}(\log\log s)^\gamma \quad (s \to \infty), \tag{11}$$

and the measure $d\mu(x) = \chi_{(1,2)}\exp(-1/(2-x))\,dx$ yields

$$F(s) \simeq \frac{s^2}{(\log s)^{3/4}}\exp(-2\sqrt{\log s}) \quad (s \to \infty). \tag{12}$$

Note that these two asymptotic estimates are not trivial but require results from [4]. Thus, if F satisfies (11) or (12) then (3) is true for all $a \in L^1$. Clearly, the restriction to $\alpha > 1$ in (10) and (11) can be dropped since $F(s) = O(s)$ for $\alpha \le 1$.

Finally, if $\mu((\alpha,\beta)) > 0$ for some interval $(\alpha,\beta) \subset (1,\infty)$, then, for $s > 1$,

$$F(s) \ge \int_\alpha^\beta s^x\,d\mu(x) \ge s^\alpha \mu((\alpha,\beta)),$$

which is impossible if $F(s) = O(s(\log s)^\gamma)$ with $\gamma \in \mathbb{R}$ as $s \to \infty$. Consequently, Proposition 2.5 does not give an answer to Problem 1.2.

Remark 2.7. To prove Corollary 2.6 we used that $\int_1^\infty s^x\delta(x-p)dx = s^p$ for $p > 1$. The formula $\int_1^\infty s^x\delta'(x-p)dx = -s^p\log s$ $(p > 1)$ is perhaps a reasonable starting point for an analysis that yields (3) for $F(s) = s^p\log s$ $(p > 1)$ and all $a \in L^1$. Note that again $F(s) = s\log s$ would remain unattained.

3. Determinants of banded Hermitian Toeplitz matrices

While Section 2 was concerned with test functions that do not satisfy the usual growth restrictions at infinity, we now turn to Szegö's formula (1) for $a \in L^\infty$ and $F(\lambda) = \log|\lambda|$. In that case the behavior of $F(\lambda)$ as $|\lambda| \to \infty$ is not of importance. The delicacy comes rather from the singularity of the function at the origin.

For $F(\lambda) = \log|\lambda|$, formula (1) can be written in the form

$$\frac{1}{n}\log|D_n(a)| = \int_{-\pi}^{\pi}\log|a(\theta)|\,\frac{d\theta}{2\pi} + o(1), \tag{13}$$

where $D_n(a) = \det T_n(a)$. If $|a(\theta)| \geq \varepsilon > 0$ for almost all $\theta \in (-\pi, \pi)$, then $|\lambda_j(T_n(a))| \geq \varepsilon > 0$ for all j and n and (13) is known to be true (see, e.g., [5]). We are interested in the question on what happens to (13) if $|a|$ is not bounded away from zero.

Things are fairly transparent for tridiagonal Toeplitz matrices. If $a(\theta) = e^{i\theta} + e^{-i\theta} = 2\cos\theta$, straightforward computation gives

$$D_n(a) = \begin{cases} 0 & \text{if} \quad n \equiv 1,3 \pmod 4 \\ -1 & \text{if} \quad n \equiv 2 \pmod 4 \\ 1 & \text{if} \quad n \equiv 0 \pmod 4 \end{cases} \qquad \text{and} \qquad \int_{-\pi}^{\pi} \log|a(\theta)| \frac{d\theta}{2\pi} = 0.$$

Thus, (13) is true if (and only if) we agree to define $\log 0 := 0$. Without that agreement, (13) would still hold with the restriction to even n's. The general form of a real-valued trigonometric polynomial of degree 1 is

$$a(\theta) = c + be^{i\theta} + \overline{b}e^{-i\theta} \tag{14}$$

where $c \in \mathbb{R}$ and $b \in \mathbb{C} \setminus \{0\}$. This function can also be written in the form $a(\theta) = c + 2|b|\cos(\theta - \theta_0)$. The eigenvalues of $T_n(a)$ are

$$\lambda_j := \lambda_j(T_n(a)) = c + 2|b|\cos\frac{\pi j}{n+1} \quad (j = 1, \ldots, n) \tag{15}$$

(see [2] or [5]). They densely fill the segment $[c - 2|b|, c + 2|b|]$ as $n \to \infty$. This segment contains the origin, or equivalently, the function a has a real zero, if and only if $|c| \leq 2|b|$, in which case there is a unique $x \in [0,1]$ such that

$$c + 2|b|\cos\pi x = 0. \tag{16}$$

The degree of approximation of x by rational fractions with n in the denominator is measured by

$$\psi_x(n) := n \min_{j=1,\ldots,n-1} \left| x - \frac{j}{n} \right|.$$

Throughout what follows we define $\log 0 := 0$.

Proposition 3.1. *Let a be given by* (14)*, suppose $|c| \leq 2|b|$, and define $x \in [0,1]$ by* (16)*. Then*

$$\frac{1}{n}\log|D_n(a)| = \int_0^{2\pi} \log|a(\theta)| \frac{d\theta}{2\pi} + \frac{1}{n}\log\psi_x(n+1) + o(1). \tag{17}$$

Proof. Let $j_n \in \{1, \ldots, n\}$ satisfy $\left| x - \frac{j_n}{n+1} \right| = \frac{\psi_x(n+1)}{n+1}$. From (15) and (16) we infer that, as $n \to \infty$,

$$\begin{aligned} |\lambda_{j_n}| &= 2|b| \left| \cos\frac{\pi j_n}{n+1} - \cos\pi x \right| \sim 2\pi|b|\sin(\pi x) \left| \frac{j_n}{n+1} - x \right| \\ &= 2\pi|b|\sin(\pi x)\, \frac{\psi_x(n+1)}{n+1}, \end{aligned}$$

where $\alpha_n \sim \beta_n$ means that $\alpha_n/\beta_n \to 1$. Hence,

$$\frac{1}{n}\log|\lambda_{j_n}| = \frac{1}{n}\log\frac{\psi_x(n+1)}{n+1} + o(1) = \frac{1}{n}\log\psi_x(n+1) + o(1).$$

Since $|\lambda_j - x| \ge \frac{1}{2(n+1)}$ for $j \ne j_n$, it follows from [9, No. 29 on p. 53] that

$$\begin{aligned}\frac{1}{n}\sum_{j\ne j_n}\log|\lambda_j| &= \frac{1}{\pi}\int_0^\pi \log|c+2|b|\cos\theta|d\theta + o(1)\\ &= \frac{1}{2\pi}\int_0^{2\pi}\log|c+2|b|\cos\theta|d\theta + o(1),\end{aligned}$$

which completes the proof. □

Thus, (13) holds if and only if $\frac{1}{n}\log\psi_x(n+1) \to 0$ as $n \to \infty$.

Proposition 3.2. *The set $\mathcal{X}$ of all $x \in [0,1]$ for which $\frac{1}{n}\log\psi_x(n+1)$ does not converge to zero as $n \to \infty$ is uncountable, dense, and of measure zero. Rational numbers do not belong to $\mathcal{X}$.*

Proof. We use results that can all be found in [6] and [7]. Clearly, $\frac{1}{n}\log\psi_x(n+1)$ goes to 0 if and only if $\frac{1}{n}\log\psi_x(n) \to 0$. If $x = p/q$ is rational, then $\psi_x(n) = 0$ if n is divisible by q and $\psi_x(n) \ge 1/(qn)$ if n is not divisible by q. Thus, $\frac{1}{n}\log\psi_x(n) \to 0$ and $x \notin \mathcal{X}$. So assume x is irrational.

A function $\varphi : \mathbb{N} \to (0,1]$ is called an approximation function for an irrational number y if there exists a sequence $n_k \to \infty$ such that $\psi_y(n_k) < \varphi(n_k)$. Let $A(\varphi)$ denote the set of all irrational $y \in (0,1)$ for which φ is an approximation function.

Let the irrational number x be in $\mathcal{X}$. Then there exist an $\varepsilon > 0$ and a sequence $n_k \to \infty$ such that $\frac{1}{n_k}|\log\psi_x(n_k)| \ge 2\varepsilon$ for all n_k. Thus, letting $\mathcal{X}_N$ denote the set of all irrational $y \in \mathcal{X}$ for which there is a sequence $n_k \to \infty$ such that $\frac{1}{n_k}|\log\psi_y(n_k)| \ge 2/N$ for all n_k, we have $\mathcal{X} \subset \cup_{N=1}^\infty \mathcal{X}_N$. If $x \in \mathcal{X}_N$, then $|\log\psi_x(n_k)| \ge 2n_k/N$ for some sequence $n_k \to \infty$ and hence we obtain that $\psi_x(n_k) \le e^{-2n_k/N} < e^{-n_k/N}$. This implies that $x \in A(\varphi_N)$ for $\varphi_N(n) = e^{-n/N}$. Since $\sum_{n=1}^\infty \varphi_N(n) < \infty$, the set $A(\varphi_N)$ has measure zero (Khinchin's theorem). Consequently, $\mathcal{X}_N \subset A(\varphi_N)$ is of measure zero and thus $\mathcal{X}$ also has measure zero.

Finally, take $\varphi(n) = e^{-n^2}$. If $x \in A(\varphi)$, then $\psi_x(n_k) < e^{-n_k^2}$ for some sequence $n_k \to \infty$. For this sequence, $\frac{1}{n_k}|\log\psi_x(n_k)| \to \infty$. Thus, $x \in \mathcal{X}$. We have proved that $A(\varphi) \subset \mathcal{X}$. Since $A(\varphi)$ is known to be uncountable and dense for every function $\varphi : \mathbb{N} \to (0,1]$, it follows that $\mathcal{X}$ is uncountable and dense as well. □

We remark that irrational numbers that are algebraic over $\mathbb{Q}$ do also not belong to $\mathcal{X}$ since, by a theorem of Liouville, $\psi_x(n) \ge cn^{1-\alpha}$ with some $c > 0$ if x is algebraic of degree $\alpha \ge 2$.

The example $a(\theta) = 2\cos\theta$ considered in the beginning motivates the restriction to matrix dimensions n that belong to arithmetic progressions. Here is a result in this direction.

Proposition 3.3. *Let a be given by (14), suppose $|c| \le 2|b|$, and define $x \in [0,1]$ by (16). For each natural number $\ell \ge 1$, there exists a subsequence $\{n_1, n_2, n_3, \dots\}$ of $\{\ell, 2\ell, 3\ell, \dots\}$ such that $\frac{1}{n_k}\log\psi_x(n_k+1) \to 0$ and hence*

$$\frac{1}{n_k}\log|D_{n_k}(a)| = \int_{-\pi}^{\pi}\log|a(\theta)|\,\frac{d\theta}{2\pi} + o(1) \quad (k \to \infty).$$

Proof. Since $\frac{1}{n}\log\psi_x(n+1) \to 0$ for rational numbers x, we may assume that x is irrational. Let $j_{m\ell+1} \in \{1, \dots, m\ell\}$ be the number for which $|(m\ell+1)x - j_{m\ell+1}| = \psi_x(m\ell+1)$. We have $\psi_x(m\ell+1) < 1$ and hence $\log\psi_x(m\ell+1) < 0$ for all m. Assume there is no sequence $m_k \to \infty$ such that $\frac{1}{m_k\ell}\log\psi_x(m_k\ell+1) \to 0$. Then there is an $\varepsilon > 0$ such that $\frac{1}{m\ell}\log\psi_x(m\ell+1) \le -\varepsilon$ and thus $\psi_x(m\ell+1) \le e^{-m\ell\varepsilon}$ for all m. We write x in the base $\ell+1$:

$$x = \frac{x_1}{\ell+1} + \frac{x_2}{(\ell+1)^2} + \frac{x_3}{(\ell+1)^3} + \cdots .$$

Then

$$(\ell+1)^k x = x_1(\ell+1)^{k-1} + \cdots + x_k + \frac{x_{k+1}}{\ell+1} + \frac{x_{k+2}}{(\ell+1)^2} + \cdots$$

and hence

$$j_{(\ell+1)^k} = x_1(\ell+1)^{k-1} + \cdots + x_k \quad \text{or} \quad j_{(\ell+1)^k} = x_1(\ell+1)^{k-1} + \cdots + x_k + 1.$$

If $1 \le x_{k+1} \le \ell - 1$, then

$$\psi_x((\ell+1)^k) = \left|(\ell+1)^k x - j_{(\ell+1)^k}\right| \ge \frac{1}{\ell+1}.$$

Consequently, $\frac{1}{\ell+1} \le e^{-[(\ell+1)^k-1]\varepsilon}$, which is impossible for $k \ge k_1$. It follows that $x_{k+1} = 0$ or $x_{k+1} = \ell$ for all $k \ge k_1$. Suppose we have $x_{k+1} = 0$ and $x_{k+2} = \ell$. In that case $j_{(\ell+1)^k} = x_1(\ell+1)^k + \cdots + x_k$ and

$$\begin{aligned}\psi_x((\ell+1)^k) &= (\ell+1)^k x_k - j_{(\ell+1)^k} \\ &= \frac{\ell}{(\ell+1)^2} + \frac{x_{k+3}}{(\ell+1)^3} + \frac{x_{k+4}}{(\ell+1)^4} + \cdots \ge \frac{\ell}{(\ell+1)^2}.\end{aligned}$$

As $\frac{\ell}{(\ell+1)^2} \le e^{-[(\ell+1)^k-1]\varepsilon}$ is not true whenever $k \ge k_2$, we conclude that the combination $x_{k+1} = 0$ and $x_{k+2} = \ell$ is not possible for $k \ge k_2$. Thus, either $x_k = 0$ for all $k \ge k_3$ or $x_k = \ell$ for all $k \ge k_3$. But this is a contradiction to our hypothesis that x be irrational. □

The previous three propositions dealt with trigonometric polynomials of the degree 1. We don't know a useful result on general trigonometric polynomials, but we can say at least the following. For convenience we assume that a is nonconstant and $\|a\|_\infty := \max|a(\theta)| = 1$; the general case can be reduced to this case by multiplying a by an appropriate constant. Our assumption guarantees that all eigenvalues of $T_n(a)$ lie in $(-1,1)$. We denote by $\|T_n^+(a)\|$ the spectral norm of the Moore-Penrose inverse of $T_n(a)$. Clearly, $\|T_n^+(a)\|$ is nothing but the maximum of the inverses of the absolute values of the nonzero eigenvalues of $T_n(a)$. Thus, $\|T_n^+(a)\| > 1$ for all $n \geq 1$.

Proposition 3.4. *Let a be a real-valued trigonometric polynomial with at least one real zero and with $\|a\|_\infty = 1$. If $\alpha \geq 1$ is the maximal order of the real zeros of a, then*

$$\frac{1}{n}\log|D_n(a)| = \int_{-\pi}^{\pi}\log|a(\theta)|\frac{d\theta}{2\pi} + O\left(\frac{\log\|T_n^+(a)\|}{n^{1/\alpha}}\right) + o(1).$$

Proof. We omit the technical details and confine ourselves to an outline of the basic steps. Let $r \geq 1$ be the degree of the trigonometric polynomial a. We denote by $C_{n+R}(a)$ the circulant matrix of order $n+R$ that can be associated with the banded Toeplitz matrix $T_n(a)$ (see, e.g., [2, p. 33]). Let $\tilde{\lambda}_1 \leq \cdots \leq \tilde{\lambda}_{n+R}$ be the eigenvalues of $C_{n+R}(a)$ and denote by $\lambda_1 \leq \cdots \leq \lambda_n$ the eigenvalues of $T_n(a)$. By Cauchy's interlacing theorem, $\tilde{\lambda}_j \leq \lambda_j \leq \tilde{\lambda}_{j+R}$ for $j = 1, \ldots, n$. The eigenvalues of $C_{n+R}(a)$ are

$$a\left(\frac{2\pi k}{n+R}\right) \quad (k = 0, \ldots, n+R-1).$$

Since $|a'(\theta)|$ is bounded, we have

$$\left|a\left(\frac{2\pi(k+1)}{n+R}\right) - a\left(\frac{2\pi k}{n+R}\right)\right| = |a'(\theta_k)|\,\frac{2\pi}{n+R} \leq \frac{M}{n+R}$$

with some constant $M < \infty$ for all k. We can show that

$$\#\left\{j : |\lambda_j| < \frac{3M}{n+R}\right\} = O(n^{1-1/\alpha}), \tag{18}$$

where $\#E$ denotes the number of elements of a finite set E. Let $F(\lambda) = \log|\lambda|$. Using (18) we get

$$\left|\sum_{|\lambda_j| < \frac{3M}{n+R}} \frac{F(\lambda_j)}{n}\right| = O\left(\frac{\log\|T_n^+(a)\|}{n^{1/\alpha}}\right).$$

Furthermore, we can prove that

$$\sum_{\lambda_j \leq -\frac{3M}{n+R}} \frac{|F(\lambda_j) - F(\tilde{\lambda}_j)|}{n} + \sum_{\lambda_j \geq \frac{3M}{n+R}} \frac{|F(\lambda_j) - F(\tilde{\lambda}_{j+R})|}{n} = o(1).$$

Making use of [9, No. 29 on p. 53] we finally obtain that

$$\sum_{\lambda_j \leq -\frac{3M}{n+R}} \frac{F(\tilde{\lambda}_j)}{n} + \sum_{\lambda_j \geq \frac{3M}{n+R}} \frac{F(\tilde{\lambda}_{j+R})}{n} = \int_{-\pi}^{\pi}\log|a(\theta)|\,\frac{d\theta}{2\pi} + o(1).$$

Putting the things together we arrive at the asserted formula. □

Proposition 3.4 tells us that (13) is certainly true if $\|T_n^+(a)\|$ increases at most polynomially. If, in particular, $a(\theta) \geq 0$ for all θ or $a(\theta) \leq 0$ for all θ, then $\|T_n^+(a)\| = \|T_n^{-1}(a)\| = O(n^\alpha)$ (see [2, Corollary 4.34]) and hence (13) is valid.

The following problem (which is Problem 1.1 in the case $\ell = 1$) is perhaps the simplest one should tackle to gain more insight into the matter.

Problem 3.5. *Let a be a real-valued trigonometric polynomial that assumes both positive and negative values and let $\ell \geq 1$ be a natural number. Is there a subsequence $\{n_1, n_2, n_3, \ldots\}$ of $\{\ell, 2\ell, 3\ell, \ldots\}$ such that*

$$\frac{1}{n_k} \log |D_{n_k}(a)| = \int_{-\pi}^{\pi} \log |a(\theta)| \frac{d\theta}{2\pi} + o(1) \quad (k \to \infty) \ ?$$

4. Higher dimensions and block case

The problems considered so far are of even greater interest for higher-dimensional block Toeplitz operators. Let $a : (-\pi, \pi)^d \to \mathbb{C}^{N \times N}$ be in L^1 on $(-\pi, \pi)^d$ and put

$$a_j = \frac{1}{(2\pi)^d} \int_{(-\pi,\pi)^d} a(\theta_1, \ldots, \theta_d) e^{-i(j_1\theta_1 + \cdots + i_d\theta_d)} \, d\theta_1 \ldots d\theta_d$$

for $j = (j_1, \ldots, j_d) \in \mathbb{Z}^d$. We denote by $T_n(a)$ the operator acting on the ℓ^2 space of all functions $\varphi : \{1, \ldots, n\}^d \to \mathbb{C}^N$ by the rule

$$(T_n(a)\varphi)_j = \sum_{k \in \{1,\ldots,n\}^d} a_{j-k}\varphi_k, \quad j \in \{1, \ldots, n\}^d.$$

After preliminary work by many authors, Tilli [12] found textbook proofs of the following formulas: if $a = a^*$ then

$$\lim_{n\to\infty} \frac{1}{n^d} \sum_{j=1}^{Nn^d} F(\lambda_j(T_n(a))) = \int_{(-\pi,\pi)^d} \sum_{k=1}^{N} F(\lambda_k(a(\theta))) \frac{d\theta}{(2\pi)^d}$$

and for general a we have

$$\lim_{n\to\infty} \frac{1}{n^d} \sum_{j=1}^{Nn^d} F(s_j(T_n(a))) = \int_{(-\pi,\pi)^d} \sum_{k=1}^{N} F(s_k(a(\theta))) \frac{d\theta}{(2\pi)^d}$$

provided that $F : \mathbb{R} \to \mathbb{R}$ is bounded and uniformly continuous.

It follows in particular that if $a : (-\pi, \pi)^d \to \mathbb{C}^{N \times N}$ is a matrix-valued trigonometric polynomial in d variables such that $a(\theta)$ is a positive definite Hermitian matrix for every $\theta \in (-\pi, \pi)^d$, then

$$\frac{1}{n^d} \log |D_n(a)| = \int_{(-\pi,\pi)^d} \log |\det a(\theta)| \frac{d\theta}{(2\pi)^d} + o(1). \tag{19}$$

Now suppose

$$a(\theta_1, \theta_2) = w + ue^{i\theta_1} + u^*e^{-i\theta_1} + ve^{i\theta_2} + v^*e^{-i\theta_2} \tag{20}$$

where $v \in \mathbb{C}^{N \times N}$ is Hermitian and u, v are arbitrary matrices in $\mathbb{C}^{N \times N}$. Then $a = a^*$. From Section 3 we know that we cannot expect (19) to be true for indefinite matrix functions. Here is a precise result in this direction.

Proposition 4.1. *Let $\ell \ge 1$ be a natural number. There exist scalar-valued functions of the form* (20) *and a subsequence $\{n_1, n_2, n_3, \dots\}$ of $\{\ell, 2\ell, 3\ell, \dots\}$ such that $\log|a|$ is in L^1 but*

$$\lim_{k\to\infty} \frac{1}{n_k^2} \log|D_{n_k}(a)| = \infty.$$

Proof. Let

$$a(\theta_1,\theta_2) = a_1(\theta_1)a_2(\theta_2) = (c_1 + b_1 e^{i\theta_1} + \overline{b}_1 e^{-i\theta_1})(c_2 + b_2 e^{i\theta_2} + \overline{b}_2 e^{-i\theta_2})$$

with $c_j \in \mathbb{R}$ and $b_j \in \mathbb{C}$. Clearly, $\log|a| \in L^1$. Suppose $|c_j| \le 2|b_j|$ and define $x_j \in [0,1]$ by $c_j + 2|b_j|\cos\pi x_j = 0$. We have $T_n(a) = T_n(a_1) \otimes T_n(a_2)$ and hence $D_n(a) = D_n(a_1)^n D_n(a_2)^n$. Thus,

$$\frac{1}{n^2}\log|D_n(a)| = \frac{1}{n}\log|D_n(a_1)| + \frac{1}{n}\log|D_n(a_2)|.$$

Taking into account Proposition 2.1 and the identity

$$\int_{-\pi}^{\pi} \log|a_1(\theta_1)| \frac{d\theta_1}{2\pi} + \int_{-\pi}^{\pi} \log|a_2(\theta_2)| \frac{d\theta_2}{2\pi} = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \log|a(\theta_1,\theta_2)| \frac{d\theta_1 d\theta_2}{(2\pi)^2}$$

we therefore see that $\frac{1}{n^2}\log|D_n(a)|$ equals

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \log|a(\theta_1,\theta_2)| \frac{d\theta_1 d\theta_2}{(2\pi)^2} + \frac{1}{n}\log\psi_{x_1}(n+1) + \frac{1}{n}\log\psi_{x_2}(n+1) + o(1).$$

We are so left with showing that there are $x \in [0,1]$ and $m_1 < m_2 < \cdots$ such that $\frac{1}{m_k}\log\psi_x(\ell m_k + 1) \to \infty$. But if x is a number whose representation in the base $\ell+1$ is of the form $x = 0.10\dots010\dots010\dots010\dots$ with sufficiently long chains of zeros, then $\frac{1}{m_k}\log\psi_x(\ell m_k+1) \to \infty$ if the numbers $\ell m_k + 1$ are chosen as appropriate powers of $\ell + 1$. □

We conclude with two open questions. The first of them can perhaps be tackled as in the proof of Proposition 3.4 by comparing Toeplitz matrices with appropriate circulants. The second of these questions seems to be harder and is in fact the thing one really wants to know.

Problem 4.2. *Let $a : (-\pi,\pi]^d \to \mathbb{C}^{N\times N}$ be a trigonometric matrix polynomial such that $a(\theta)$ is a Hermitian matrix with all eigenvalues in $(-1,1)$ for every $\theta \in (-\pi,\pi]^d$. Suppose also that $\det a$ has at least one zero in $(-\pi,\pi]^d$. Is there a number $\beta \in (0,\infty)$ such that*

$$\frac{1}{n^d}\log|D_n(a)| = \frac{1}{2\pi}\int_{(-\pi,\pi)^d} \log|\det a(\theta)| \frac{d\theta}{(2\pi)^d} + O\left(\frac{\log\|T_n^+(a)\|}{n^\beta}\right) + o(1) \ ?$$

Problem 4.3. *Let a be of the form* (20) *and let $\ell \ge 1$ be a natural number. Is there a subsequence $\{n_1, n_2, n_3, \dots\}$ of $\{\ell, 2\ell, 3\ell, \dots\}$ such that $D_{n_k}(a) \ne 0$ for all k and*

$$\frac{1}{n_k^2}\log|D_{n_k}(a)| = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \log|\det a(\theta_1,\theta_2)| \frac{d\theta_1 d\theta_2}{(2\pi)^2} + o(1)$$

as $k \to \infty$?

Added in proof. Problems 1.2 and 2.4 were recently solved in [3]. The answer to Problem 1.2 is in the affirmative, while that to Problem 2.4 is negative.

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