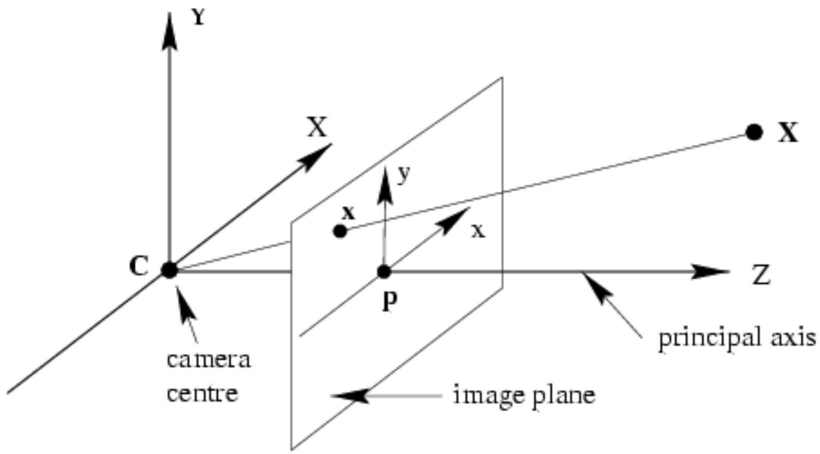
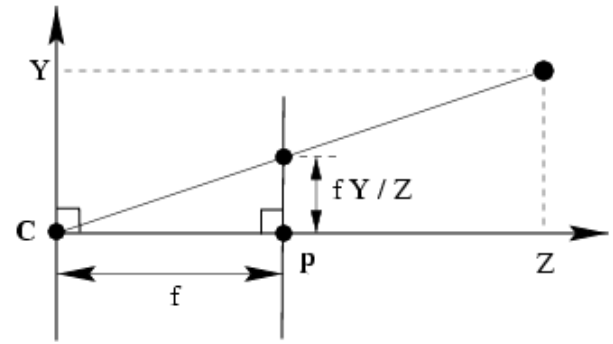
Camera Models
Chapter 6

Multiple View Geometry

Pinhole camera model

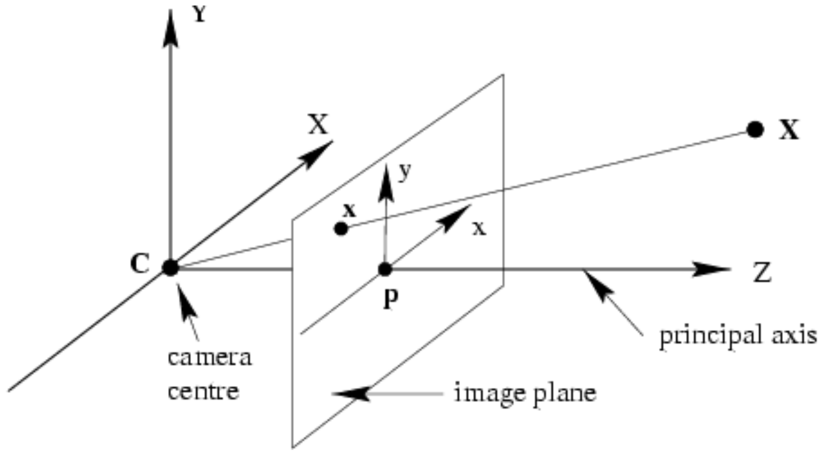


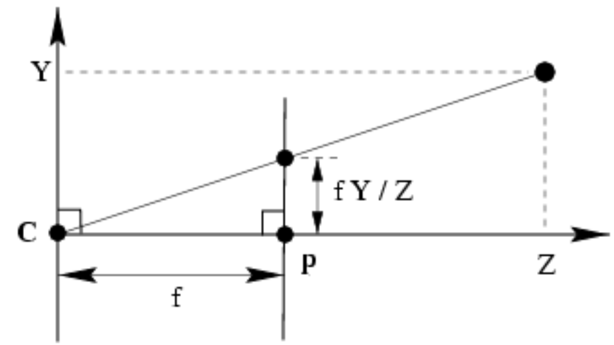


$$(X, Y, Z)^T \mapsto (fX / Z, fY / Z)^T$$

$$\begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} fX \\ fY \\ Z \end{pmatrix} = \begin{bmatrix} f & & & 0 \\ & f & & 0 \\ & & 1 & 0 \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}$$

Pinhole camera model

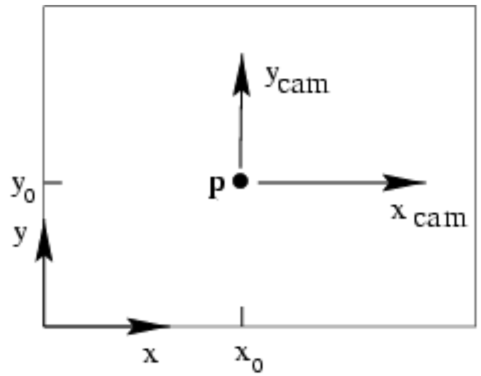




$$\begin{pmatrix} fX \\ fY \\ Z \end{pmatrix} = \begin{bmatrix} f & & & 0 \\ & f & & 0 \\ & & 1 & 0 \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}$$

$$\mathrm{x} = \mathrm{PX} \qquad \mathrm{P} = \mathrm{diag}(f, f, 1)\left[\mathrm{I} \mid 0\right]$$

Principal point offset

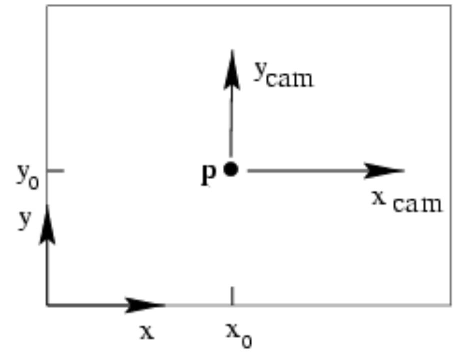


$$(X,Y,Z)^T \mapsto (fX/Z + p_x, fY/Z + p_y)^T$$

$(p_x, p_y)^T$ principal point

$$\begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} fX + Zp_x \\ fY + Zp_x \\ Z \end{pmatrix} = \begin{bmatrix} f & & p_x & 0 \\ & f & p_y & 0 \\ & & 1 & 0 \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}$$

Principal point offset

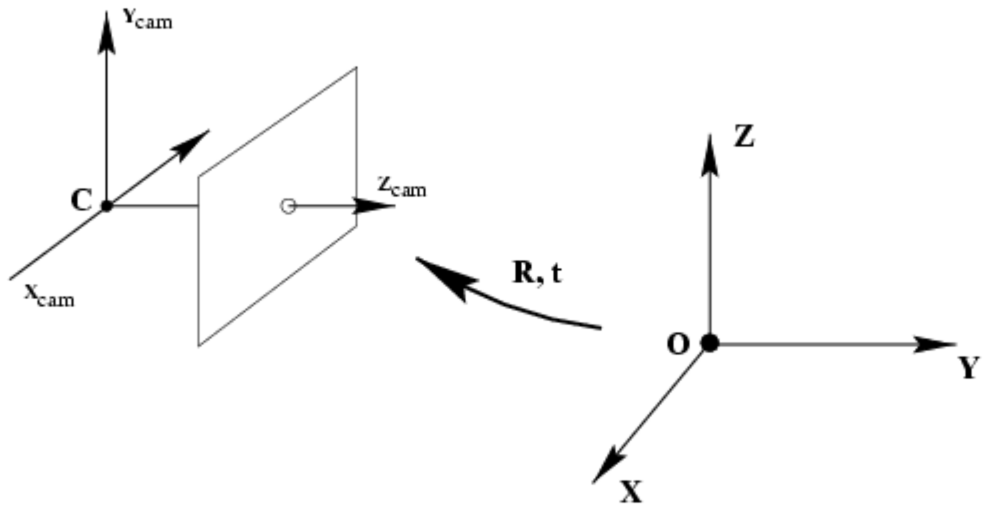


$$\begin{pmatrix} fX + Zp_x \\ fY + Zp_x \\ Z \end{pmatrix} = \begin{bmatrix} f & & p_x & 0 \\ & f & p_y & 0 \\ & & 1 & 0 \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}$$

$$\mathrm{x} = \mathrm{K}\left[\mathrm{I} \mid 0\right]\mathrm{X}_{\mathrm{cam}}$$

$$K = \begin{bmatrix} f & & p_x \\ & f & p_y \\ & & 1 \end{bmatrix}$$ calibration matrix

Camera rotation and translation



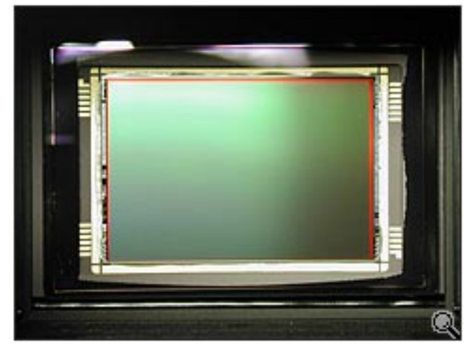
$$\tilde{X}_{cam} = R(\tilde{X} - \tilde{C})$$

$$X_{cam} = \begin{bmatrix} R & -R\tilde{C} \\ 0 & 1 \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} = \begin{bmatrix} R & -R\tilde{C} \\ 0 & 1 \end{bmatrix} X$$

$$x = K[I \mid 0] X_{cam} \qquad x = KR[I \mid -\tilde{C}] X$$

$$x = PX \qquad P = K[R \mid t] \qquad t = -R\tilde{C}$$

CCD camera



$$K = \begin{bmatrix} \alpha_x & & p_x \\ & \alpha_x & p_y \\ & & 1 \end{bmatrix}$$

$$\alpha_x = f m_x$$
$$\alpha_y = f m_y$$

m_x and m_y are number of pixels in unit distance in x and y directions.

Finite projective camera

$$K = \begin{bmatrix} \alpha_x & s & p_x \\ & \alpha_x & p_y \\ & & 1 \end{bmatrix}$$

$$\mathrm{P} = \underbrace{\mathrm{KR}}\left[\mathrm{I} \mid -\tilde{\mathrm{C}}\right]$$ 11 dof (5+3+3)

non-singular

decompose P in K,R,C?

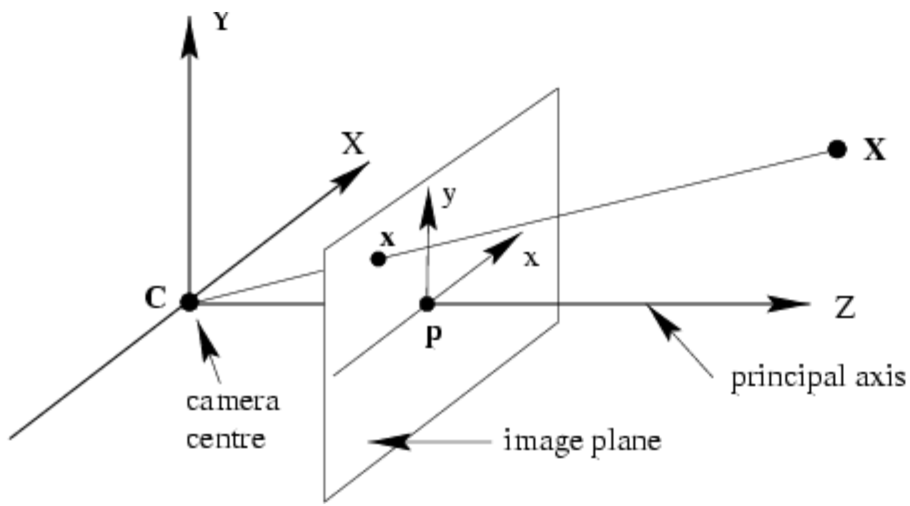
$$\mathrm{P} = \left[\mathrm{M} \mid \mathrm{p}_4\right] \quad \left[\mathrm{K}, \mathrm{R}\right] = RQ(\mathrm{M}) \quad \tilde{\mathrm{C}} = -\mathrm{M}^{-1}\mathrm{p}_4$$

{finite cameras}={P_{3x4} | det M≠0}

If rank P=3, but rank M<3, then cam at infinity

Camera anatomy

Camera center
Column points
Principal plane
Axis plane
Principal point
Principal ray



Camera center

Null-space camera projection matrix

$$\mathrm{PC} = 0$$

$$\mathrm{X} = \lambda \mathrm{A} + (1-\lambda)\mathrm{C}$$

$$\mathrm{x} = \mathrm{PX} = \lambda \mathrm{PA} + (1-\lambda)\mathrm{PC}$$

For any A, all points on AC are projected on the image of A, therefore C is camera center

Image of camera center is $(0,0,0)^T$, i.e. undefined

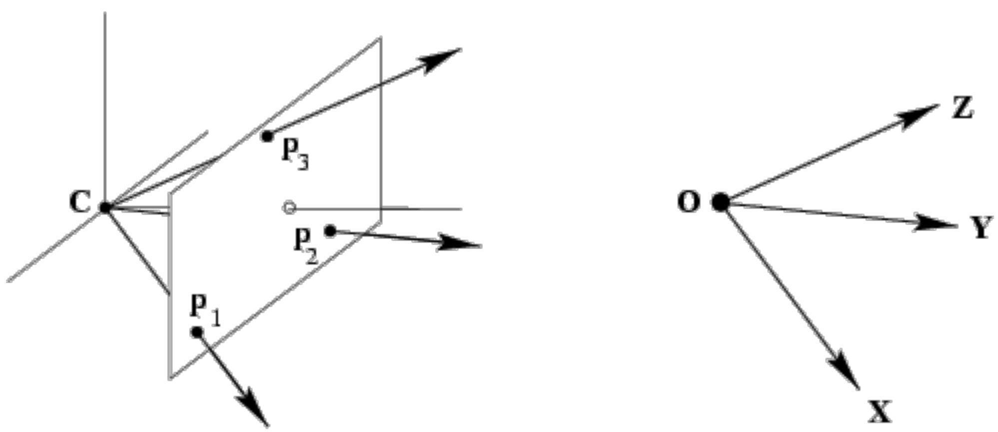
Finite cameras: $C = \begin{pmatrix} -\mathrm{M}^{-1}\mathrm{p}_4 \\ 1 \end{pmatrix}$

Infinite cameras: $C = \begin{pmatrix} \mathrm{d} \\ 0 \end{pmatrix}, \mathrm{Md} = 0$

Column vectors

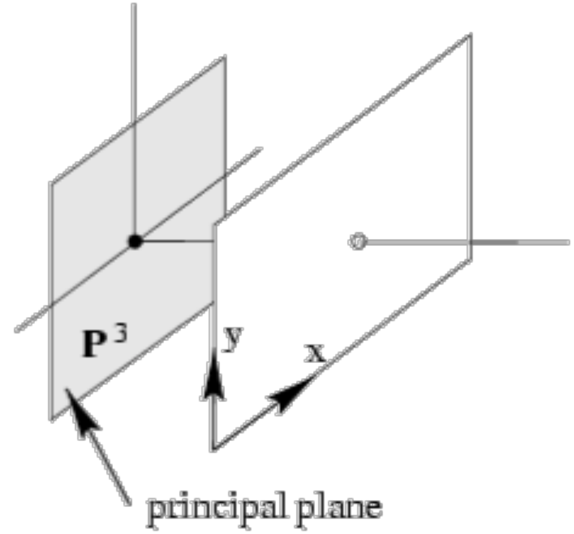
$$[\mathrm{p}_2] = [\mathrm{p}_1 \mathrm{p}_2 \mathrm{p}_3 \mathrm{p}_4]\begin{bmatrix}0\\1\\0\\0\end{bmatrix}$$

Image points corresponding to X,Y,Z directions and origin

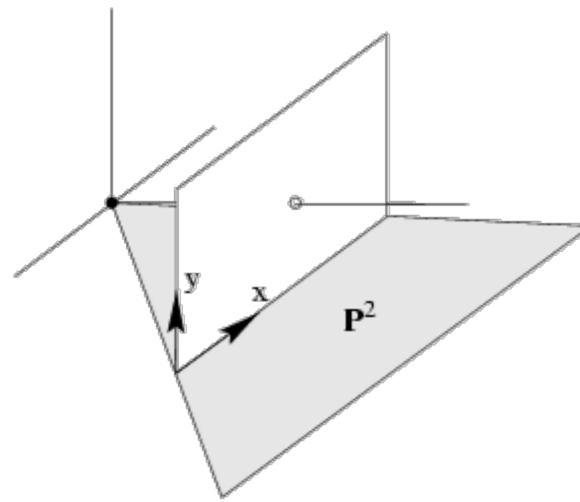


Row vectors

$$\begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = \begin{bmatrix} \mathrm{p}^{1\mathsf{T}} \\ \mathrm{p}^{2\mathsf{T}} \\ \mathrm{p}^{3\mathsf{T}} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

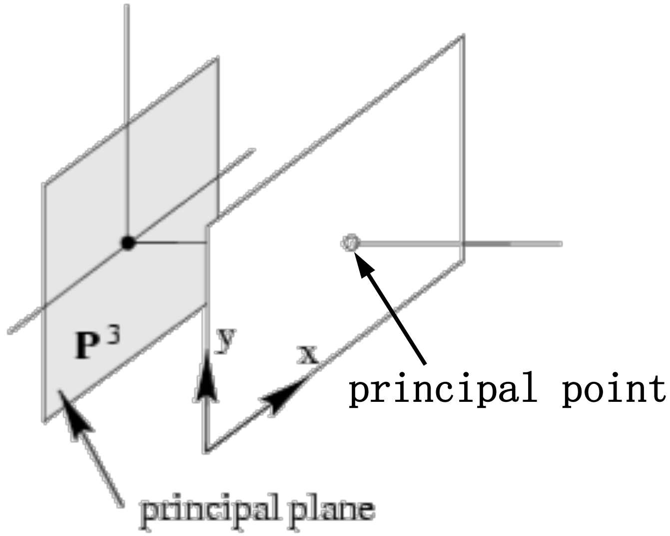


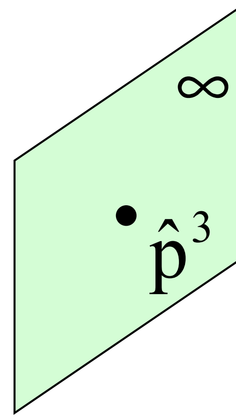
$$\begin{bmatrix} 0 \\ y \\ w \end{bmatrix} = \begin{bmatrix} \mathrm{p}^{1\mathsf{T}} \\ \mathrm{p}^{2\mathsf{T}} \\ \mathrm{p}^{3\mathsf{T}} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$



note: p^1,p^2 dependent on image reparametrization

The principal point



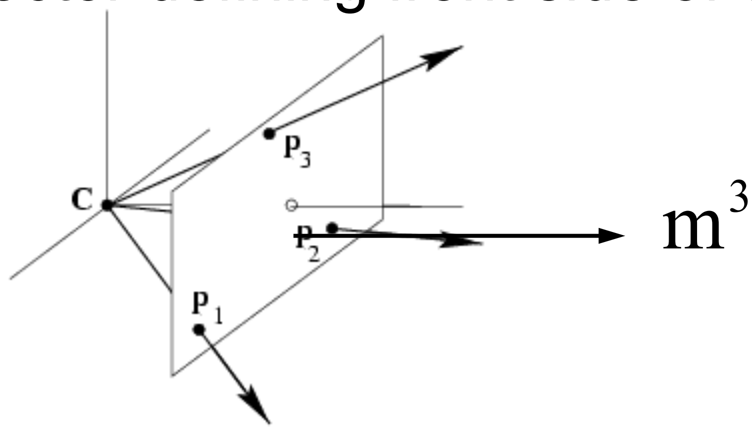


$$\hat{\mathrm{p}}^3 = (p_{31}, p_{32}, p_{33}, 0)$$

$$\mathrm{x}_0 = \mathrm{P}\hat{\mathrm{p}}^3 = \mathrm{M}\mathrm{m}^3$$

The principal axis vector

vector defining front side of camera



$$x = P_{cam} X_{cam} = K[I \mid 0] X_{cam}$$

$$v = \det(M) m^3 = (0,0,1)^T$$

$$P_{cam} \mapsto k P_{cam}$$

$$v \mapsto k^4 v$$

(direction unaffected)

$$P = kKR[I \mid -\tilde{C}] = [M \mid p_4]$$

$$P_{cam} \mapsto k P_{cam}$$

$$v \mapsto \det(kM) k m^3 = k^4 v$$

because $\det(R) > 0$

Action of projective camera on point

Forward projection

$$x = PX$$

$$x = PD = [M \mid p_4]D = Md$$

Back-projection

$$PC = 0$$

$$X = P^{+}x \qquad P^{+} = P^{\mathsf{T}}\left(PP^{\mathsf{T}}\right)^{-1} \qquad PP^{+} = I$$

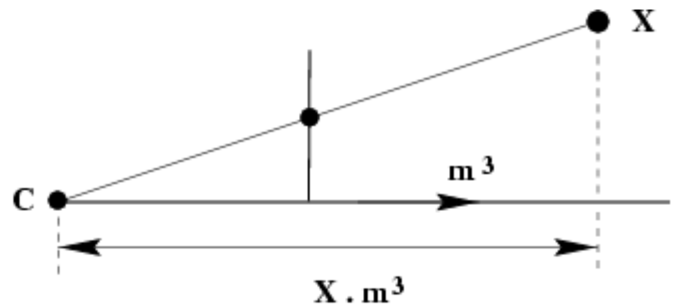
(pseudo-inverse)

$$X(\lambda) = P^{+}x + \lambda C$$

$$d = M^{-1}x$$

$$X(\lambda) = \mu\begin{pmatrix} M^{-1}x \\ 0 \end{pmatrix} + \begin{pmatrix} -M^{-1}p_4 \\ 1 \end{pmatrix} = \begin{pmatrix} M^{-1}(\mu x - p_4) \\ 1 \end{pmatrix}$$

Depth of points



$$w = P^{3^T} X = P^{3^T}(X - C) = m^{3^T}(\tilde{X} - \tilde{C})$$

(PC=0) (dot product)

If $\det M > 0; \|m^3\| = 1$,
then m^3 unit vector in positive direction

$$\text{depth}(X; P) = \frac{\text{sign}(\det M)\, w}{T \|m^3\|}$$

$$X = (X, Y, Z, T)^T$$

Camera matrix decomposition

Finding the camera center

$$\mathrm{PC} = 0$$ (use SVD to find null-space)

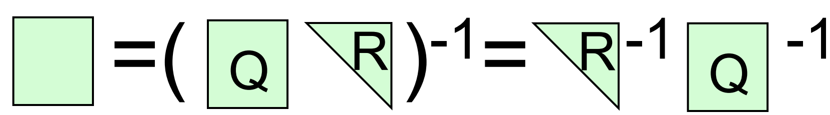
$$X = \det([\mathrm{p}_2, \mathrm{p}_3, \mathrm{p}_4]) \quad Y = -\det([\mathrm{p}_1, \mathrm{p}_3, \mathrm{p}_4])$$

$$Z = \det([\mathrm{p}_1, \mathrm{p}_2, \mathrm{p}_4]) \quad T = -\det([\mathrm{p}_1, \mathrm{p}_2, \mathrm{p}_3])$$

Finding the camera orientation and internal parameters

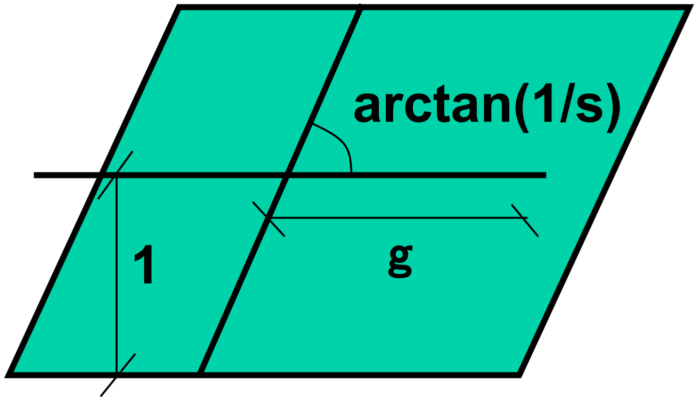
$$\mathrm{M} = \mathrm{KR}$$ (use RQ decomposition ~QR)

(if only QR, invert)



When is skew non-zero?

$$K = \begin{bmatrix} \alpha_x & s & p_x \\ & \alpha_x & p_y \\ & & 1 \end{bmatrix}$$



for CCD/CMOS, always s=0

Image from image, s≠0 possible
(non coinciding principal axis)

resulting camera: HP

Euclidean vs. projective

general projective interpretation

$$P = [3\times3\ \text{homography}]\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}[4\times4\ \text{homography}]$$

Meaningfull decomposition in K,R,t requires Euclidean image and space

Camera center is still valid in projective space

Principal plane requires affine image and space

Principal ray requires affine image and Euclidean space

Cameras at infinity

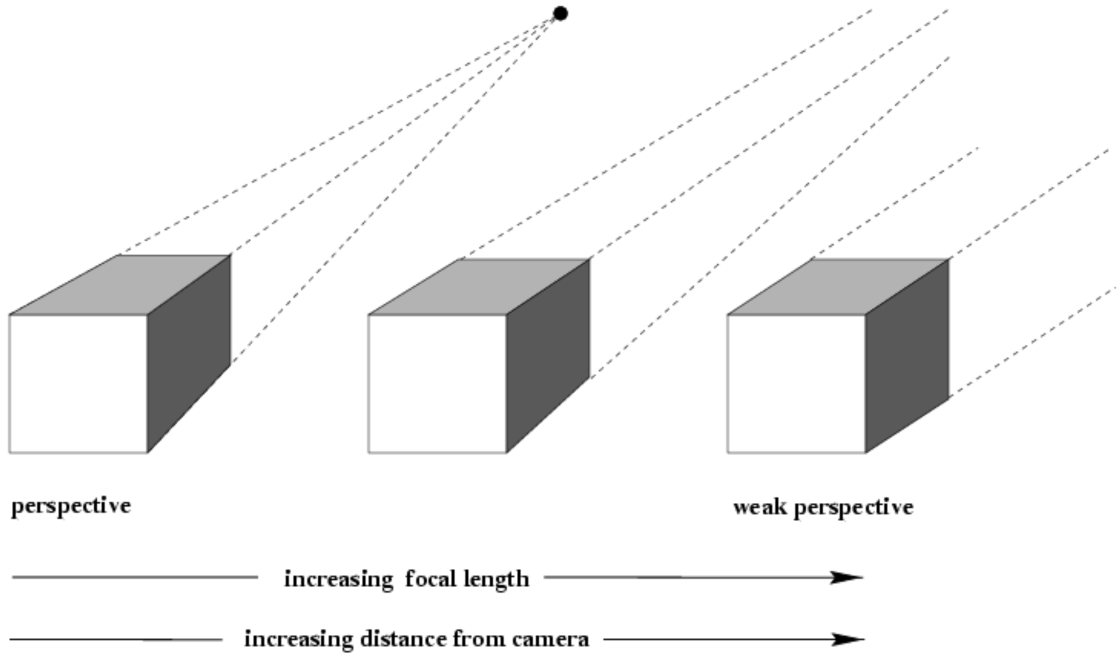
Camera center at infinity

$$P\begin{bmatrix} d \\ 0 \end{bmatrix} = 0 \Rightarrow \det M = 0$$

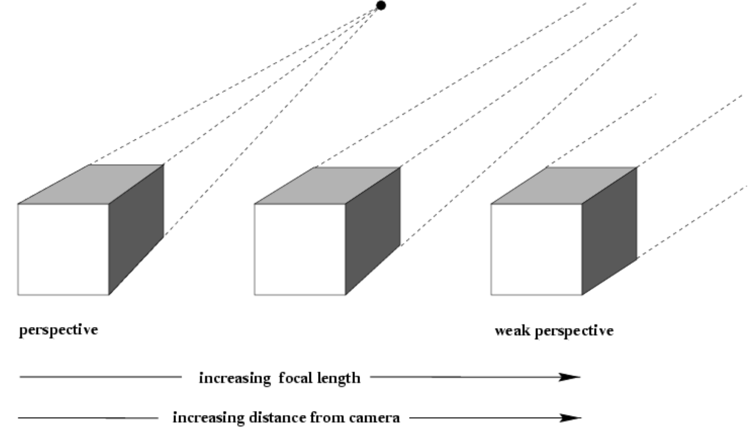
Affine and non-affine cameras

<u>Definition</u>: affine camera has $P^{3T}=(0,0,0,1)$

Affine cameras



Affine cameras



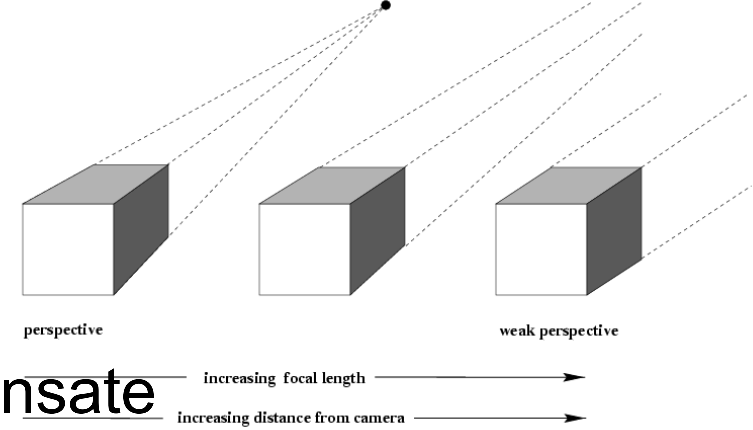
$$P_0 = KR\left[I \mid -\tilde{C}\right] = K\begin{bmatrix} r^{1T} & -r^{1T}\tilde{C} \\ r^{2T} & -r^{2T}\tilde{C} \\ r^{3T} & -r^{3T}\tilde{C} \end{bmatrix}$$

$$d_0 = -r^{3T}\tilde{C}$$

$$P_t = K\begin{bmatrix} r^{1T} & -r^{1T}\left(\tilde{C} - t r^3\right) \\ r^{2T} & -r^{2T}\left(\tilde{C} - t r^3\right) \\ r^{3T} & -r^{3T}\left(\tilde{C} - t r^3\right) \end{bmatrix} = K\begin{bmatrix} r^{1T} & -r^{1T}\tilde{C} \\ r^{2T} & -r^{2T}\tilde{C} \\ r^{3T} & d_t \end{bmatrix}$$

modifying p_{34} corresponds to moving along principal ray

Affine cameras



now adjust zoom to compensate

$$P_t = K \begin{bmatrix} d_t / d_0 & & \\ & d_t / d_0 & \\ & & 1 \end{bmatrix} \begin{bmatrix} r^{1T} & -r^{1T}\tilde{C} \\ r^{2T} & -r^{2T}\tilde{C} \\ r^{3T} & d_t \end{bmatrix}$$

$$= \frac{d_t}{d_0} K \begin{bmatrix} r^{1T} & -r^{1T}\tilde{C} \\ r^{2T} & -r^{2T}\tilde{C} \\ r^{3T} d_t / d_0 & d_0 \end{bmatrix}$$

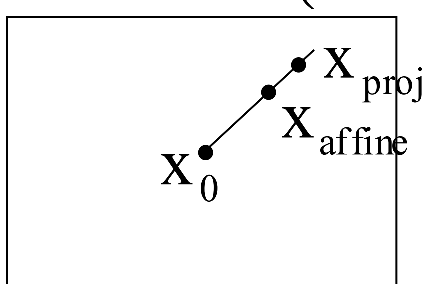
$$P_\infty = \lim_{t \to \infty} P_t = K \begin{bmatrix} r^{1T} & -r^{1T}\tilde{C} \\ r^{2T} & -r^{2T}\tilde{C} \\ 0 & d_0 \end{bmatrix}$$

Error in employing affine cameras

$$X = \begin{pmatrix} \alpha r^1 + \beta r^2 \\ 1 \end{pmatrix}$$ point on plane parallel with principal plane and through origin, then $P_0 X = P_t X = P_\infty X$

$$X = \begin{pmatrix} \alpha r^1 + \beta r^2 + \Delta r^3 \\ 1 \end{pmatrix}$$ general points

$$x_{proj} = P_0 X = K \begin{pmatrix} \tilde{x} \\ \tilde{y} \\ d_0 + \Delta \end{pmatrix} \qquad x_{affine} = P_\infty X = K \begin{pmatrix} \tilde{x} \\ \tilde{y} \\ d_0 \end{pmatrix}$$



Affine imaging conditions

$$x_{affine} - x_{proj} = \frac{\Delta}{d_0}\left(x_{proj} - x_0\right)$$

Approximation should only cause small error

1. D much smaller than d_0
2. Points close to principal point
   (i.e. small field of view)

Decomposition of P_∞

$$P_\infty = \begin{bmatrix} K_{2x2} & \tilde{x}_0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{R} & \tilde{t} \\ 0 & d_0 \end{bmatrix} = \begin{bmatrix} d_0^{-1} K_{2x2} & \tilde{x}_0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{R} & \tilde{t} \\ 0 & 1 \end{bmatrix}$$

absorb d_0 in K_{2x2}

$$= \begin{bmatrix} K_{2x2}\tilde{R} & K_{2x2}\tilde{t} + \tilde{x}_0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} K_{2x2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{R} & \tilde{t} + K_{2x2}^{-1}\tilde{x}_0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} K_{2x2} & K_{2x2}\tilde{t} + \tilde{x}_0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{R} & 0 \\ 0 & 1 \end{bmatrix}$$

$$P_\infty = \begin{bmatrix} K_{2x2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{R} & \tilde{t} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} K_{2x2} & \tilde{x}_0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{R} & 0 \\ 0 & 1 \end{bmatrix}$$

alternatives, because 8dof (3+3+2), not more

Summary parallel projection

$$P_\infty = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

canonical representation

$$K = \begin{bmatrix} K_{2\times2} & 0 \\ 0 & 1 \end{bmatrix}$$

calibration matrix

principal point is not defined

A hierarchy of affine cameras

$$P_\infty = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad H = \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix}$$

Orthographic projection

$$P_\infty = \begin{bmatrix} r^{1T} & t_1 \\ r^{1T} & t_2 \\ 0 & 1 \end{bmatrix}$$ (5dof)

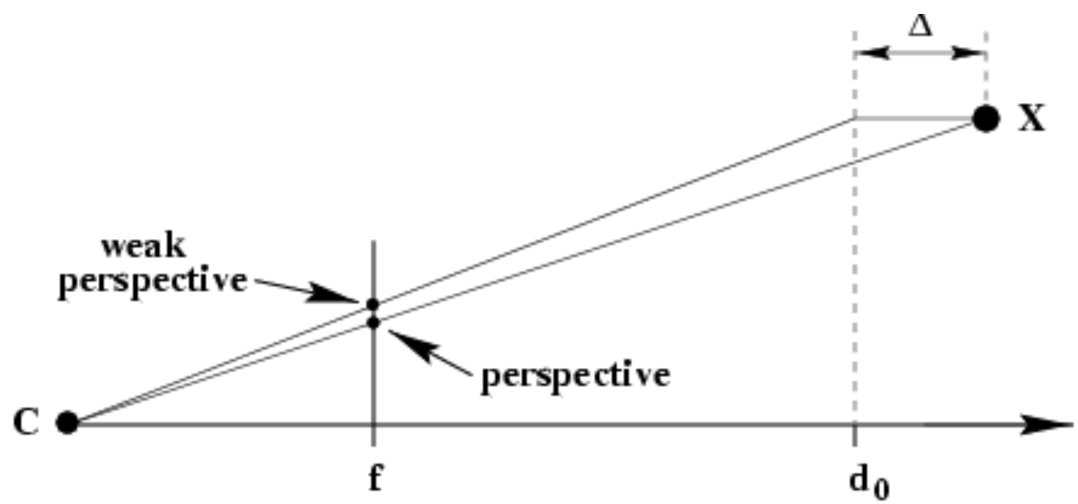
Scaled orthographic projection

$$P_\infty = \begin{bmatrix} r^{1T} & t_1 \\ r^{1T} & t_2 \\ 0 & 1/k \end{bmatrix}$$ (6dof)

A hierarchy of affine cameras

Weak perspective projection

$$P_\infty = \begin{bmatrix} \alpha_x & & \\ & \alpha_y & \\ & & 1 \end{bmatrix} \begin{bmatrix} \mathrm{r}^{1\mathrm{T}} & t_1 \\ \mathrm{r}^{1\mathrm{T}} & t_2 \\ 0 & 1/k \end{bmatrix} \quad \text{(7dof)}$$



A hierarchy of affine cameras

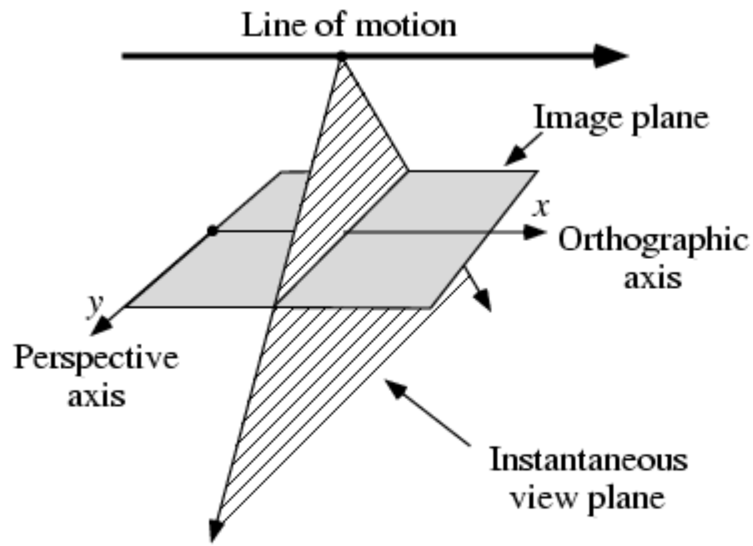
Affine camera (8dof)

$$P_A = \begin{bmatrix} \alpha_x & s & \\ & \alpha_y & \\ & & 1 \end{bmatrix} \begin{bmatrix} r^{1T} & t_1 \\ r^{1T} & t_2 \\ 0 & 1/k \end{bmatrix} \quad P_A = \begin{bmatrix} m_{11} & m_{12} & m_{13} & t_1 \\ m_{21} & m_{22} & m_{23} & t_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P_A = [3\times 3 \text{ affine}] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} [4\times 4 \text{ affine}]$$

1. Affine camera=camera with principal plane coinciding with P_∞
2. Affine camera maps parallel lines to parallel lines
3. No center of projection, but direction of projection $P_A D=0$ (point on P_∞)

Pushbroom cameras



(11dof)

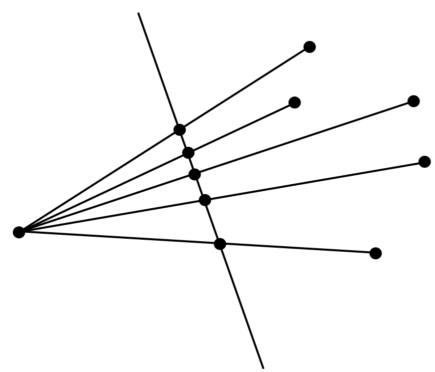
$$\mathrm{X} = (X, Y, X, T)^{\mathrm{T}} \qquad \mathrm{PX} = (x, y, w)^{\mathrm{T}} \qquad (x, y/w)^{\mathrm{T}}$$

Straight lines are not mapped to straight lines!
(otherwise it would be a projective camera)

Line cameras

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

(5dof)



Null-space PC=0 yields camera center

Also decomposition $P_{2\times3} = K_{2\times2} R_{2\times2} [I_{2\times2} \mid -\tilde{c}]$