

Lattice-based Robust Distributed Source Coding

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Abstract—In this paper, we propose a lattice-based robust distributed source coding system for two correlated sources and provide a detailed performance analysis under the high resolution assumption. It is shown, among other things, that, in the asymptotic regime where 1) the side distortion approaches 0 and 2) the ratio between the central and side distortions approaches 0, our scheme is capable of achieving the information-theoretic limit of quadratic multiple description coding when the two sources are identical, whereas a variant of the random coding scheme by Chen and Berger with Gaussian codes has a performance loss of 0.5 bits relative to this limit.

Index Terms—Distributed source coding, lattice quantization, high resolution analysis.

I. INTRODUCTION

Distributed source coding is a crucial category of source coding problems, which has received significant attention over the past few decades. In distributed source coding, multiple correlated sources are encoded separately and sent to a central decoder for joint decoding. For the case when the central decoder is required to recover both sources losslessly, Slepian and Wolf [1] characterized the achievable rate region. The case when one source is available as side information at the decoder, while the other source may be recovered with some distortion, was solved by Wyner and Ziv [2]. A general formulation of the distributed source coding problem in the lossy case was provided by Berger [3] and Tung [4]. However, the solution has been found only in certain special cases [5]–[10].

A closely related problem is the CEO problem introduced in [11], where the correlated sources are noisy observations of a single remote source, whose reconstruction is required at the joint decoder. The rate-distortion region for this problem has been completely characterized in the quadratic Gaussian case by Oohama [12] and Prabhakaran *et al.* [13].

Most of past work assume that the central decoder receives the information sent by all separate encoders. However, in practice this may not be true. For instance, in the case of wireless communications, the quality of the channels may be fluctuating. If the channel connecting some encoder with the fusion centre becomes very bad, the decoder is no longer able to recover the transmitted information. In such cases a robust system is desired. The robust version of the distributed source coding problem was considered in the CEO setting by Ishwar *al.* [14] and Chen and Berger [15]. The design of practical schemes was addressed in [16]–[18], where iterative algorithms were employed for locally optimal designs. On the other hand, the work of Heegard and Berger [19] considers

the robust version of the Wyner-Ziv problem and provides a characterization of the rate-distortion region.

The robust distributed source coding (RDSC) problem for the case of two correlated sources is considered in this paper. We propose a structured coding scheme based on lattices and provide a detailed performance analysis under the high resolution assumption. Note that when the two sources are identical, the setting being considered coincides with that of the classical multiple description coding (MDC) problem [20]–[29]. For this case, our analysis indicates that, in the asymptotic regime where 1) the side distortion approaches 0 and 2) the ratio between the central and side distortions approaches 0, the proposed lattice-based scheme is capable of achieving the information-theoretic limit of quadratic MDC. For comparison we consider a variant of the random coding scheme originally proposed by Chen and Berger [15] for the robust CEO problem and prove that the sum-rate of the latter system with Gaussian codes is 0.5 bits higher than the sum-rate of our proposed approach in the same asymptotic regime.

Our design is inspired by the prior work on multiple description lattice vector quantizers (MDLVQ) of Vaishampayan *et al.* [30] and Huang and Wu [31]. It is worth pointing out that lattices have been used in prior work in other distributed source coding problems [33]–[36]. Most of the aforementioned papers use dithered lattice quantization, except for the work of Servetto [34], which performs the analysis under the assumption of very high rate and very high correlation.

The paper is structured as follows. Section II presents the formulation of the RDSC problem. In Section III we analyze the performance of a random-coding-based RDSC scheme (similar to the one proposed in [15]) with Gaussian codes and prove that it does not achieve the information-theoretic limit of quadratic MDC in the asymptotic regime where the side distortion and the ratio between the central and side distortions approach 0. Section IV introduces definitions and notations related to lattices. Section V presents the main results of this work, namely the asymptotic performance analysis of the proposed lattice-based RDSC scheme. It is shown, among other things, that our design is able to achieve the fundamental limit of quadratic MDC in the aforementioned asymptotic regime. Section VI presents the detailed operation of the proposed lattice-based RDSC scheme. Finally, Section VII concludes the paper.

II. PROBLEM FORMULATION

Consider two sources X_1 and X_2 with joint probability distribution $f_{X_1 X_2}$. The two sources generate a jointly i.i.d. random process $(X_{1,k}, X_{2,k})_{k \in \mathbb{N}}$. We will consider an RDSC system as illustrated in Figure 1. The system consists of two encoders and three decoders. Encoder i , $i = 1, 2$, has access

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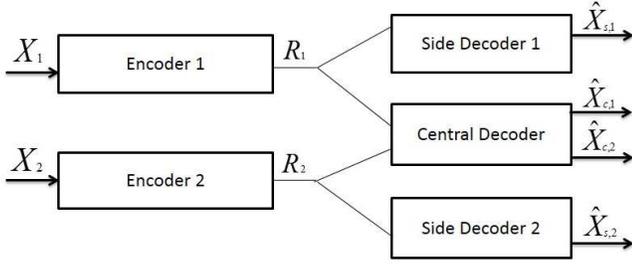


Fig. 1. Block diagram of robust distributed source coding.

only to source X_i , while the side decoder i receives only the information sent by encoder i and aims at reconstructing source X_i , $i = 1, 2$. The central decoder receives the information from both encoders and aims at reconstructing both sources X_1 and X_2 .

For each $i = 1, 2$, let $d_i : \mathcal{X}_i \times \hat{\mathcal{X}}_i \rightarrow [0, \infty)$ be a distortion measure, where \mathcal{X}_i and $\hat{\mathcal{X}}_i$ are the source alphabet and the reconstruction alphabet for source X_i , respectively. The distortion measures are extended to sequences of length n as follows

$$d_i(x_i^n, \hat{x}_i^n) = \frac{1}{n} \sum_{k=1}^n d_i(x_{i,k}, \hat{x}_{i,k}),$$

where $x_i^n = (x_{i,1}, \dots, x_{i,n})$, $\hat{x}_i^n = (\hat{x}_{i,1}, \dots, \hat{x}_{i,n})$.

A six-tuple $(R_1, R_2, d_{s,1}, d_{s,2}, d_{c,1}, d_{c,2})$ is said achievable, if for any $\epsilon > 0$ and all sufficiently large n , there exist encoding functions

$$f_i^{(n)} : \mathcal{X}_i^n \rightarrow \{1, 2, \dots, \lfloor 2^{n(R_i + \epsilon)} \rfloor\}, \quad i = 1, 2,$$

and decoding functions

$$\begin{aligned} g_{s,i}^{(n)} : \{1, 2, \dots, \lfloor 2^{n(R_i + \epsilon)} \rfloor\} &\rightarrow \hat{\mathcal{X}}_i^n, \quad i = 1, 2, \\ g_{c,i}^{(n)} : \{1, 2, \dots, \lfloor 2^{n(R_1 + \epsilon)} \rfloor\} \times \{1, 2, \dots, \lfloor 2^{n(R_2 + \epsilon)} \rfloor\} &\rightarrow \hat{\mathcal{X}}_i^n, \\ &i = 1, 2, \end{aligned}$$

such that

$$\mathbb{E} \left[d_i(X_i^n, \hat{X}_{t,i}^n) \right] \leq d_{t,i} + \epsilon, \quad i = 1, 2, \quad t = s, c,$$

where $\mathbb{E}[\cdot]$ denotes the expectation operator and

$$\hat{X}_{t,i}^n = g_{t,i}^{(n)}(f_i^{(n)}(X_i^n)), \quad i = 1, 2, \quad t = s, c.$$

The RDSC rate-distortion region, denoted by \mathcal{RD} , is the set of all such achievable six-tuples.

Furthermore, if Y is a random variable over some discrete alphabet \mathcal{Y} , with probability mass function p_Y , and $\sum_{y \in \mathcal{Y}} p_Y(y) \log_2 p_Y(y)$ is finite, then the entropy of Y is $H(Y) \triangleq -\sum_{y \in \mathcal{Y}} p_Y(y) \log_2 p_Y(y)$. If $X^n \in \mathbb{R}^n$ is a continuous random vector with probability density function (pdf) f_{X^n} , and the quantity $\int_{\mathbb{R}^n} f_{X^n}(x^n) \log_2 f_{X^n}(x^n) dx^n$ is finite, then the differential entropy of X^n is $h(X^n) \triangleq -\int_{\mathbb{R}^n} f_{X^n}(x^n) \log_2 f_{X^n}(x^n) dx^n$.

III. A RANDOM-CODING-BASED RDSC SCHEME

In this section, we adapt a random coding scheme originally proposed by Chen and Berger [15] for the robust CEO problem to the current setting and analyze the asymptotic performance of this scheme when specialized to the MDC scenario.

Theorem 1. We have $\mathcal{RD}_{in} \subseteq \mathcal{RD}$, where \mathcal{RD}_{in} denotes the set of rate-distortion tuples $(R_1, R_2, d_{s,1}, d_{s,2}, d_{c,1}, d_{c,2})$ for which there exist auxiliary random variables U_1, U_2, W_1, W_2 (jointly distributed with the generic source variables X_1 and X_2) satisfying the following Markov chain

$$W_1 \leftrightarrow U_1 \leftrightarrow X_1 \leftrightarrow X_2 \leftrightarrow U_2 \leftrightarrow W_2, \quad (1)$$

and deterministic mappings $g_{s,i} : W_i \rightarrow \hat{\mathcal{X}}_i$, $g_{c,i} : U_1 \times U_2 \rightarrow \hat{\mathcal{X}}_i$, $i = 1, 2$, such that

$$R_1 \geq I(X_1; W_1) + I(X_1; U_1 | U_2, W_1, W_2),$$

$$R_2 \geq I(X_2; W_2) + I(X_2; U_2 | U_1, W_1, W_2),$$

$$R_1 + R_2 \geq I(X_1; W_1) + I(X_2; W_2)$$

$$+ I(X_1, X_2; U_1, U_2 | W_1, W_2),$$

$$d_{s,i} \geq \mathbb{E}[d_i(X_i, g_{s,i}(W_i))], \quad i = 1, 2, \quad (2)$$

$$d_{c,i} \geq \mathbb{E}[d_i(X_i, g_{c,i}(U_1, U_2))], \quad i = 1, 2. \quad (3)$$

The inner bound \mathcal{RD}_{in} in Theorem 1 is achievable by the following random coding scheme. Roughly speaking, encoder i produces (W_i, U_i) , where W_i is a (lossy) description of X_i , and U_i is a refinement of W_i , $i = 1, 2$. Moreover, W_i is encoded using the conventional lossy source code while U_i is encoded using the Berger-Tung code [3], [4] with (W_1, W_2) as the decoder side information, $i = 1, 2$. Side decoder i can recover W_i and use $g_{s,i}(W_i)$ as an estimate of X_i , $i = 1, 2$. The central decoder can recover (U_1, U_2) (as well as (W_1, W_2)) and use $g_{c,i}(U_1, U_2)$ as an estimate of X_i , $i = 1, 2$. The proof of Theorem 1 is similar to [15, Theorem 1] and is thus omitted.

In the rest of this paper, we assume $\mathcal{X}_1 = \mathcal{X}_2 = \hat{\mathcal{X}}_1 = \hat{\mathcal{X}}_2 = \mathbb{R}$ and adopt the squared distance as the distortion measure unless specified otherwise. To facilitate the evaluation of the achievable rate-distortion tuples in Theorem 1, we shall focus on so-called Gaussian codes (in the sense of [37]), which correspond to the following construction. Let

$$U_i = X_i + Z_i, W_i = U_i + Z'_i, \quad i = 1, 2, \quad (4)$$

where Z_1, Z_2, Z'_1, Z'_2 are zero-mean mutually independent Gaussian random variables and are independent of (X_1, X_2) . It is clear that U_1, U_2, W_1, W_2 constructed according to (4) satisfy the Markov chain condition (1). Moreover, we restrict $g_{s,i}$ and $g_{c,i}$, $i = 1, 2$, to be linear MMSE estimators; as such, (2) and (3) can be rewritten as

$$d_{s,i} \geq \text{LMMSE}(X_i | W_i), \quad i = 1, 2, \quad (5)$$

$$d_{c,i} \geq \text{LMMSE}(X_i | U_1, U_2), \quad i = 1, 2, \quad (6)$$

where LMMSE denotes the squared distortion induced by the linear MMSE estimate.

Now consider the special case where $X_1 = X_2 = X$, $d_{s,1} = d_{s,2} = d_s$, and $d_{c,1} = d_{c,2} = d_c$. This is exactly the setting of the symmetric MDC problem. We shall assume that the source

variable X is of mean zero, variance σ_X^2 , and finite differential entropy $h(X)$. It is well-known (see, e.g., [37], [38]) that in the asymptotic regime

$$d_s \rightarrow 0, \frac{d_c}{d_s} \rightarrow 0, \quad (7)$$

the minimum sum-rate of symmetric MDC is given by

$$R_{MD}(d_s, d_c) = 2h(X) - \frac{1}{2} \log_2(4(2\pi e)^2 d_s d_c) + o(1). \quad (8)$$

We shall show that in the same asymptotic regime the minimum sum-rate of the random-coding-based RDSC scheme in Theorem 1 with Gaussian codes as defined by (4)–(6) is given by

$$R_{RC}(d_s, d_c) = 2h(X) - \frac{1}{2} \log_2(2(2\pi e)^2 d_s d_c) + o(1), \quad (9)$$

therefore is 0.5 bits away from the fundamental limit.

First note that in the current setting (5) and (6) can be written equivalently as

$$d_s \geq \left(\frac{1}{\sigma_X^2} + \frac{1}{\sigma_{Z_i}^2 + \sigma_{Z'_i}^2} \right)^{-1}, \quad i = 1, 2, \quad (10)$$

$$d_c \geq \left(\frac{1}{\sigma_X^2} + \frac{1}{\sigma_{Z_1}^2} + \frac{1}{\sigma_{Z_2}^2} \right)^{-1}, \quad (11)$$

which implies

$$\sigma_{Z_i}^2 + \sigma_{Z'_i}^2 \leq (1 + o(1))d_s, \quad i = 1, 2, \quad (12)$$

$$\frac{\sigma_{Z_1}^2 \sigma_{Z_2}^2}{\sigma_{Z_1}^2 + \sigma_{Z_2}^2} \leq (1 + o(1))d_c, \quad (13)$$

in the asymptotic regime (7). It can be verified that

$$\begin{aligned} & I(X; W_1) + I(X; W_2) + I(X; U_1, U_2 | W_1, W_2) \\ &= I(X; W_1) + I(X; W_2) + I(X; U_1, U_2) - I(X; W_1, W_2) \\ &= h(W_1) - h(Z_1 + Z'_1) + h(W_2) - h(Z_2 + Z'_2) + h(U_1, U_2) \\ &\quad - h(Z_1, Z_2) - h(W_1, W_2) + h(Z_1 + Z'_1, Z_2 + Z'_2) \\ &= h(W_1) + h(W_2) + h(U_1, U_2) - h(Z_1, Z_2) - h(W_1, W_2). \end{aligned} \quad (14)$$

We have

$$\begin{aligned} & h(U_1, U_2) - h(W_1, W_2) \\ &= -I(Z'_1, Z'_2; X + Z_1 + Z'_1, X + Z_2 + Z'_2) \\ &= -I(Z'_1, Z'_2; Z_1 + Z'_1 - Z_2 - Z'_2, X + Z_2 + Z'_2) \\ &= -I(Z'_1, Z'_2; Z_1 + Z'_1 - Z_2 - Z'_2) \\ &\quad - I(Z'_1, Z'_2; X + Z_2 + Z'_2 | Z_1 + Z'_1 - Z_2 - Z'_2). \end{aligned} \quad (15)$$

Substituting (15) into (14) gives

$$\begin{aligned} & I(X; W_1) + I(X; W_2) + I(X; U_1, U_2 | W_1, W_2) \\ &= h(W_1) + h(W_2) - h(Z_1, Z_2) \\ &\quad - I(Z'_1, Z'_2; Z_1 + Z'_1 - Z_2 - Z'_2) \\ &\quad - I(Z'_1, Z'_2; X + Z_2 + Z'_2 | Z_1 + Z'_1 - Z_2 - Z'_2). \end{aligned} \quad (16)$$

Note that

$$\begin{aligned} & h(Z_1, Z_2) + I(Z'_1, Z'_2; Z_1 + Z'_1 - Z_2 - Z'_2) \\ &= h(Z_1, Z_2) + h(Z_1 + Z'_1 - Z_2 - Z'_2) - h(Z_1 - Z_2) \\ &= \frac{1}{2} \log_2 \left(\frac{(2\pi e)^2 \sigma_{Z_1}^2 \sigma_{Z_2}^2 (\sigma_{Z_1}^2 + \sigma_{Z'_1}^2 + \sigma_{Z_2}^2 + \sigma_{Z'_2}^2)}{\sigma_{Z_1}^2 + \sigma_{Z_2}^2} \right) \\ &\leq \frac{1}{2} \log_2 (2(2\pi e)^2 d_s d_c) + o(1) \end{aligned} \quad (17)$$

in the asymptotic regime (7), where (17) is due to (12) and (13). Moreover,

$$\begin{aligned} & I(Z'_1, Z'_2; X + Z_2 + Z'_2 | Z_1 + Z'_1 - Z_2 - Z'_2) \\ &= h(X + Z_2 + Z'_2 | Z_1 + Z'_1 - Z_2 - Z'_2) \\ &\quad - h(X + Z_2 + Z'_2 | Z_1 + Z'_1 - Z_2 - Z'_2, Z'_1, Z'_2) \\ &= h(X + Z_2 + Z'_2 | Z_1 + Z'_1 - Z_2 - Z'_2) \\ &\quad - h(X + Z_2 | Z_1 - Z_2) \\ &= h(X + \tilde{Z}_1) - h(X + \tilde{Z}_2), \end{aligned} \quad (18)$$

where $\tilde{Z}_1 = Z_2 + Z'_2 - \mathbb{E}[Z_2 + Z'_2 | Z_1 + Z'_1 - Z_2 - Z'_2]$ and $\tilde{Z}_2 = Z_2 - \mathbb{E}[Z_2 | Z_1 - Z_2]$. It can be shown [39] that in the asymptotic regime (7)

$$h(W_i) = h(X) + o(1), \quad i = 1, 2,$$

$$h(\tilde{Z}_i) = h(X) + o(1), \quad i = 1, 2,$$

which together with (16), (17), and (18) proves that

$$\begin{aligned} & I(X; W_1) + I(X; W_2) + I(X; U_1, U_2 | W_1, W_2) \\ &\geq 2h(X) - \frac{1}{2} \log_2 (2(2\pi e)^2 d_s d_c) + o(1). \end{aligned}$$

The tightness of this lower bound can be established by choosing $\sigma_{Z_i}^2, \sigma_{Z'_i}^2, i = 1, 2$, that satisfy (10) and (11) with equalities. This completes the proof of (9).

There are two possible reasons why the performance of this random-coding-based RDSC scheme with Gaussian codes, when specialized to the symmetric MDC setting, is bounded away from the fundamental limit. Firstly, the restriction to Gaussian codes might be suboptimal. Secondly and more importantly, the random-coding-based RDSC scheme itself might be suboptimal. It is well known [20], [22] that the El Gamal-Cover (EGC) inner bound is tight for the quadratic Gaussian MDC problem. However, the inner bound \mathcal{RD}_{in} in Theorem 1, when specialized to the MDC setting, does not (at least expression-wise) coincide or subsume the EGC inner bound, therefore is unlikely to be tight. For the EGC inner bound, no Markov chain condition is imposed on the relevant auxiliary random variables. On the other hand, it is very difficult (if not impossible) to establish a single-letter inner bound of \mathcal{RD} without a Markov chain condition similar to (1). In other words, the conventional random coding argument seems to fall short of providing an RDSC scheme that does not have a performance gap when specialized to the MDC setting. This motivates us to develop an alternative RDSC scheme based on lattices that is able to close the gap in the MDC scenario.

IV. LATTICE-RELATED DEFINITIONS AND NOTATIONS

Before introducing the proposed scheme we need to clarify the lattice-related definitions and notations to be used throughout this work, which is the purpose of this section.

We will denote by x^n row vectors in \mathbb{R}^n . For $x^n = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $y^n = (y_1, \dots, y_n) \in \mathbb{R}^n$, let $\langle x^n, y^n \rangle \triangleq \sum_{i=1}^n x_i y_i$, and $\|x^n\| \triangleq \sqrt{\langle x^n, x^n \rangle}$. We will use $\mathbf{0}$ for the all-zero n -dimensional vector. For any set $\mathcal{S} \subseteq \mathbb{R}^n$, any $\sigma \in \mathbb{R}$, and any $x^n \in \mathbb{R}^n$, denote

$$x^n + \mathcal{S} \triangleq \{x^n + y^n | y^n \in \mathcal{S}\},$$

$$\sigma \mathcal{S} \triangleq \{\sigma y^n | y^n \in \mathcal{S}\}.$$

If \mathcal{S} is a measurable set then $\nu(\mathcal{S})$ denotes its volume, i.e.,

$$\nu(\mathcal{S}) \triangleq \int_{\mathcal{S}} dx^n.$$

An n -dimensional lattice Λ is the set of all possible integer linear combinations of the rows of \mathbf{G} , for some $n \times n$ non-singular matrix \mathbf{G} . In other words, we have

$$\Lambda \triangleq \{\lambda \in \mathbb{R}^n | \lambda = i^n \cdot \mathbf{G}, i^n \in \mathbb{Z}^n\}.$$

The nearest-neighbor quantizer associated with the lattice Λ is a function $Q_\Lambda(\cdot)$ which maps each $x^n \in \mathbb{R}^n$ to its nearest lattice point, i.e.,

$$Q_\Lambda(x^n) \triangleq \arg \min_{\lambda \in \Lambda} \|x^n - \lambda\|. \quad (19)$$

For every $\lambda \in \Lambda$ the set of all points mapped by Q_Λ to λ is the *Voronoi region* $V_\Lambda(\lambda)$ of λ in Λ . Note that the ties in (19) are broken in a systematic manner such that the following relation holds

$$V_\Lambda(\lambda) = \lambda + V_\Lambda(\mathbf{0}), \quad \forall \lambda \in \Lambda.$$

For any set $\mathcal{S} \subseteq \mathbb{R}^n$, let $\bar{\mathcal{S}}$ denote the closure of the set \mathcal{S} , i.e., the union of \mathcal{S} with its boundary. Then the following holds

$$\overline{V_\Lambda(\lambda)} = \{x^n \in \mathbb{R}^n | \|x^n - \lambda\| \leq \|x^n - \lambda'\| \text{ for any } \lambda' \in \Lambda\}.$$

It is worth pointing out that, according to our definition of the Voronoi region, which follows [40], not all the points on the boundary of $V_\Lambda(\lambda)$ are included in $V_\Lambda(\lambda)$, therefore $\overline{V_\Lambda(\lambda)} \neq V_\Lambda(\lambda)$. We say that two Voronoi regions $V_\Lambda(\lambda_1)$ and $V_\Lambda(\lambda_2)$, where $\lambda_1, \lambda_2 \in \Lambda$, are *adjacent* if their closures have points in common.

Further, for any $x^n \in \mathbb{R}^n$ define

$$x^n \bmod \Lambda \triangleq x^n - Q_\Lambda(x^n).$$

A *fundamental cell* of the lattice Λ is a bounded set \mathcal{C}_0 which, when shifted by the lattice points, generates a partition of \mathbb{R}^n [40]. In other words, the sets $\lambda + \mathcal{C}_0$, for all $\lambda \in \Lambda$, form a partition of \mathbb{R}^n . All measurable fundamental cells of a lattice have the same volume [40]. This value is denoted by ν_Λ and we have $\nu_\Lambda = \nu(V_\Lambda(\mathbf{0}))$. Further, for any set $\mathcal{S} \subset \mathbb{R}^n$, denote

$$\bar{r}(\mathcal{S}) \triangleq \sup_{x^n \in \mathcal{S}} \|x^n\|.$$

The open ball of radius r centered in the origin is denoted by \mathcal{B}_r , i.e.,

$$\mathcal{B}_r \triangleq \{x^n \in \mathbb{R}^n | \|x^n\| < r\}.$$

The *covering radius* of the lattice Λ is $\bar{r}_\Lambda \triangleq \bar{r}(V_\Lambda(\mathbf{0}))$. Additionally, we will denote by r_Λ the *inscribed radius* of the lattice Λ , which is defined as the radius of the largest ball centered at the origin and included in $\overline{V_\Lambda(\mathbf{0})}$.

The *normalized second moment* of a measurable set $\mathcal{S} \subseteq \mathbb{R}^n$ is defined as

$$G(\mathcal{S}) \triangleq \frac{\int_{\mathcal{S}} \|x^n\|^2 dx}{n\nu(\mathcal{S})^{\frac{2}{n}+1}}.$$

It is important to notice that the normalized second moment is invariant to scaling. The *normalized second moment of the lattice* Λ , denoted by G_Λ , is the normalized second moment of the Voronoi region of $\mathbf{0}$, i.e.,

$$G_\Lambda \triangleq G(V_\Lambda(\mathbf{0})).$$

A pair of lattices (Λ_1, Λ_2) are said to be *nested* if $\Lambda_2 \subset \Lambda_1$, i.e., if Λ_2 is a sublattice of Λ_1 . The lattice Λ_1 is termed the *fine lattice*, while Λ_2 is termed the *coarse lattice*. The index of Λ_2 with respect to Λ_1 is $N(\Lambda_2 : \Lambda_1) \triangleq \frac{\nu_{\Lambda_2}}{\nu_{\Lambda_1}}$. For any $\lambda_1 \in \Lambda_1$, the set $\lambda_1 + \Lambda_2$ is called a *coset of Λ_2 relative to Λ_1* . A set $\mathcal{F} \subset \Lambda_1$ is called a *set of coset representatives of Λ_2 relative to Λ_1* if the following two conditions hold

$$\Lambda_1 = \cup_{\lambda_1 \in \mathcal{F}} (\lambda_1 + \Lambda_2),$$

$$(\lambda_1 + \Lambda_2) \cap (\lambda'_1 + \Lambda_2) = \emptyset \text{ for any } \lambda_1 \neq \lambda'_1 \in \mathcal{F}.$$

The above conditions imply that any point $\lambda \in \Lambda_1$ can be written in a unique way as $\lambda = \lambda_1 + \lambda_2$ where $\lambda_1 \in \mathcal{F}$ and $\lambda_2 \in \Lambda_2$. As shown in [40], if \mathcal{C}_0 is a fundamental cell of the coarse lattice Λ_2 , then the set $\mathcal{C}_0 \cap \Lambda_1$ is a set of coset representatives of Λ_2 relative to Λ_1 .

We use the squared error as a distortion criterion. For any quantizer Q defined on \mathbb{R}^n and any random vector $X^n \in \mathbb{R}^n$ we denote by $D(Q, X^n)$ the per sample expected distortion, i.e.,

$$D(Q, X^n) \triangleq \frac{1}{n} \mathbb{E} [\|Q(X^n) - X^n\|^2].$$

V. MAIN RESULTS

As stated earlier in the paper, the main contribution of this work is the development of an RDSC scheme based on lattices, which is able to approach the theoretic performance limit of MDC in the asymptotic regime discussed in Section III. In this section we present the main results pertaining to the performance analysis of the proposed scheme while the details of the scheme operation are deferred to the next section.

We will assume for the rest of the paper that the marginal pdfs f_{X_1} and f_{X_2} are continuous with finite marginal differential entropies $h(X_1)$ and $h(X_2)$. We additionally assume that X_1 and X_2 have mean zero and correlation coefficient ρ .

An n -dimensional lattice robust distributed source code (LRDSC, for short) operates on input sequences of length n and is specified by a positive number r_0 and a triple of nested lattices in \mathbb{R}^n , $\mathcal{L}^{(n, r_0)} = (\Lambda_s, \Lambda_{in}, \Lambda_c)$, where $\Lambda_s \subset \Lambda_{in} \subset \Lambda_c$. The finest lattice, Λ_c , called the *central* lattice, is used for the reconstruction at the central decoder. The coarsest lattice, Λ_s , called the *side* lattice, is used for the reconstruction at the side decoders. The lattice Λ_{in} is an

auxiliary lattice used in the design; it is called the *intermediate* lattice, and is chosen such that the condition

$$r_0 + 2\bar{r}_c \leq r_{in} \quad (20)$$

is satisfied, where r_{in} denotes the inscribed radius of the lattice Λ_{in} , and \bar{r}_c denotes the covering radius of the lattice Λ_c . We point out that $\Lambda_s = \mu\Lambda_{in}$ for some even positive integer μ . The lattice $\Lambda_{s/2} \triangleq \frac{1}{2}\Lambda_s$, called the *fractional* lattice, is also used in the operation of the scheme. Note that $\Lambda_s \subset \Lambda_{s/2}$. Since μ is an even number, we also have $\Lambda_{s/2} \subset \Lambda_{in}$.

The proposed LRDCS is designed such that when the input sequences x_1^n, x_2^n are within distance r_0 from one another, the central decoder is able to refine the reconstruction of each source using the information received from the other encoder. On the other hand, when the above condition is violated, the reconstruction at the central decoder has essentially the same quality as the reconstruction at the side decoder. For this reason the probability

$$\mathcal{P}_{X_1 X_2}(r_0) \triangleq \mathbb{P}[X_2^n - X_1^n \notin \mathcal{B}_{r_0}] \quad (21)$$

plays a crucial role in the performance of the scheme. As we will see shortly, the choice of r_0 governs the trade-off between the quality of the reconstruction at the central decoder and the encoder sum-rate.

In order to evaluate the performance of the LRDCS $\mathcal{L}^{(n,r_0)}$, we assume that there are m consecutive sequences x_i^n fed to each encoder i , one at a time. The outputs corresponding to all m input sequences are further encoded losslessly. The rate and distortion of the LRDCS $\mathcal{L}^{(n,r_0)}$ are defined in the limit of m approaching ∞ . The notation $R(\mathcal{L}^{(n,r_0)})$ will be used for the sum-rate at the two encoders. Further, the notations $d_{s,i}(\mathcal{L}^{(n,r_0)})$ and $d_{c,i}(\mathcal{L}^{(n,r_0)})$ are employed for the distortions of source X_i at the side decoder i and at the central decoder, respectively, for $i = 1, 2$. We will refer to $d_{s,i}(\mathcal{L}^{(n,r_0)})$ and $d_{c,i}(\mathcal{L}^{(n,r_0)})$ as the side distortion and the central distortion of source i , respectively, for $i = 1, 2$.

In order to simplify the notations related to the lattices involved in the scheme, we will use in the sequel only the subscript $c, in, s/2$, respectively s , instead of $\Lambda_c, \Lambda_{in}, \Lambda_{s/2}$, respectively Λ_s . For instance, we will use ν_c instead of ν_{Λ_c} . Let us denote $K \triangleq N(\Lambda_{in} : \Lambda_c) = \frac{\nu_{in}}{\nu_c}$ and $M \triangleq N(\Lambda_s : \Lambda_{in}) = \frac{\nu_s}{\nu_{in}}$. Since $\Lambda_s = \mu\Lambda_{in}$, it follows that $M = \mu^n$.

In this work, we evaluate the performance of the proposed lattice-based scheme in the high resolution regime for fixed dimension n unless stated otherwise. More specifically, we require that the following relations hold simultaneously

$$M\nu_s \rightarrow 0, \quad M \rightarrow \infty, \quad K \text{ is constant.} \quad (22)$$

Note that this asymptotic regime is similar in spirit to that considered in the prior work on MDLVQ [30]–[32]. Clearly, the conditions specified in (22) imply that ν_s, ν_{in} and ν_c approach 0. Further, since Λ_{in} is a sublattice of Λ_c such that $r_0 + 2\bar{r}_c \leq r_{in}$, we also have that

$$r_0 = O(r_c) = O(\nu_c^{\frac{1}{n}})$$

as (22) holds.

In the formulations of the results in this section, we will use the statement that we have a family of LRDCSs satisfying (22). This statement means that the family is parameterized by μ and $\theta > 0$ and its members are the LRDCSs $\mathcal{L}^{(n,r_0)} = (\Lambda_s, \Lambda_{in}, \Lambda_c)$ satisfying

$$\Lambda_c = \theta\Lambda_{c,0}, \quad \Lambda_{in} = \theta\Lambda_{in,0}, \quad \Lambda_s = \mu\theta\Lambda_{in,0},$$

for some fixed lattices $\Lambda_{in,0} \subset \Lambda_{c,0}$ in \mathbb{R}^n . Then the asymptotic regime specified by (22) is equivalently stated in terms of the parameters μ and θ as follows

$$\theta \rightarrow 0, \quad \mu \rightarrow \infty, \quad \mu^2\theta \rightarrow 0.$$

Now we are ready to present the main result of this section.

Theorem 2. Consider a fixed pair of correlated sources (X_1, X_2) , a fixed positive integer n and a family of LRDCSs $\mathcal{L}^{(n,r_0)}$ satisfying (22). For $i = 1, 2$, let $U_i \triangleq Q_c(X_i^n) \bmod \Lambda_{in}$. Then in the asymptotic regime specified by (22),

$$d_{s,i}(\mathcal{L}^{(n,r_0)}) = \frac{1}{4}G_{s/2}(M\nu_s)^{\frac{2}{n}}(1 + o(1)), \quad i = 1, 2, \quad (23)$$

$$G_c\nu_c^{\frac{2}{n}}(1 + o(1)) \leq d_{c,i}(\mathcal{L}^{(n,r_0)}) \leq \frac{1}{n}\kappa_0^2\mathcal{P}_{X_1 X_2}(r_0)(M\nu_s)^{\frac{2}{n}} + G_c\nu_c^{\frac{2}{n}}(1 + o(1)), \quad i = 1, 2, \quad (24)$$

$$R(\mathcal{L}^{(n,r_0)}) = h(X_1) + h(X_2) - \frac{2}{n}\log_2 \frac{\nu_s}{K^{1/2}} + \frac{1}{n}H(U_2|U_1) + o(1), \quad (25)$$

where κ_0 is a positive constant. Additionally, we have

$$H(U_2|U_1) \leq \log_2 K, \quad (26)$$

while, if $r_0 \leq r_c$,

$$H(U_2|U_1) \leq 1 + \left(1 - \left(1 - \frac{r_0}{r_c}\right)^n + \mathcal{P}_{X_1 X_2}(r_0)\right)\log_2 K + o(1) \quad (27)$$

in the limit of (22). Furthermore, in each of relations (23)–(25) and (27), the term hidden in the little- o notation can be upperbounded by a function which does not depend on the joint pdf $f_{X_1 X_2}$ and approaches 0 under (22).

The following corollary deals with the case when $\mathcal{P}_{X_1 X_2}(r_0)$ is small enough to make the central distortion dominated by $G_c\nu_c^{\frac{2}{n}}$. In this case, the correlation coefficient between X_1 and X_2 must be close to 1, therefore we will assume that the marginal pdfs are equal.

Corollary 1. Consider a fixed pdf f_X , a fixed positive integer n and a family of LRDCSs $\mathcal{L}^{(n,r_0)}$ satisfying (22). Each LRDCS is applied to a pair of correlated sources (X_1, X_2) with marginal pdfs equal to f_X , satisfying the condition

$$\mathcal{P}_{X_1 X_2}(r_0) \leq \frac{\epsilon}{M^{\frac{4}{n}}}, \quad (28)$$

where $\lim_{(22)} \epsilon = 0$. For $i = 1, 2$, let $U_i \triangleq Q_c(X_i^n) \bmod \Lambda_{in}$. Then in the limit of (22),

$$d_{c,i}(\mathcal{L}^{(n,r_0)}) = G_c \nu_c^{\frac{2}{n}} (1 + o(1)), \quad i = 1, 2, \quad (29)$$

$$R(\mathcal{L}^{(n,r_0)}) = 2h(X) + \frac{1}{2} \log_2 \frac{G_c G_{s/2}}{4d_{s,i}(\mathcal{L}^{(n,r_0)})d_{c,i}(\mathcal{L}^{(n,r_0)})} + \frac{1}{n} H(U_2|U_1) + o(1), \quad i = 1, 2. \quad (30)$$

If, additionally, we have $\lim_{(22)} \frac{r_0}{r_c} = 0$, then

$$R(\mathcal{L}^{(n,r_0)}) = 2h(X) + \frac{1}{2} \log_2 \frac{G_c G_{s/2}}{4d_{s,i}(\mathcal{L}^{(n,r_0)})d_{c,i}(\mathcal{L}^{(n,r_0)})} + o(1). \quad (31)$$

Furthermore, in each of relations (29)-(31) the term hidden in the little- o notation can be upperbounded by a function which depends on the joint pdf $f_{X_1 X_2}$ only through $\mathcal{P}_{X_1 X_2}(r_0)$ and approaches 0 under (22).

Remark 1. Condition (28) implies that, as the limits in (22) are approached, the correlation coefficient between X_1 and X_2 approaches 1, while the marginal distributions of X_1 and X_2 remain equal to the distribution of some random variable X . This raises the question whether such a class of joint distributions exists. The answer is indeed positive as shown by the following argument. Construct first X_1 distributed as X . Then construct X_0 jointly distributed with X_1 . Further, construct X_2 jointly distributed with X_1 and X_0 such that $X_1 \leftrightarrow X_0 \leftrightarrow X_2$ form a Markov chain and the conditional distribution of X_2 given X_0 is the same as that of X_1 given X_0 . It is clear that X_1 and X_2 have the same marginal distribution, and the correlation between X_1 and X_2 can be increased by appropriately increasing the correlation between X_1 and X_0 .

Let us assume now that the marginal pdfs of X_1 and X_2 are equal to the pdf of some random variable X with variance σ_X^2 . We are interested in finding a sufficient condition on the correlation coefficient ρ under which relation (28) holds. To this end, we can apply Markov's inequality to $\|X_2^n - X_1^n\|^2$, which leads to

$$\begin{aligned} \mathcal{P}_{X_1 X_2}(r_0) &= \mathbb{P}[\|X_2^n - X_1^n\|^2 > r_0^2] \\ &< \frac{n\sigma_{X_2 - X_1}^2}{r_0^2} \\ &= \frac{2n(1 - \rho)\sigma_X^2}{r_0^2}. \end{aligned}$$

By imposing further the condition $\frac{n\sigma_{X_2 - X_1}^2}{r_0^2} \leq \frac{\epsilon}{M^{\frac{4}{n}}}$ and using the fact that $r_0 = O(\nu_c^{\frac{1}{n}})$, we obtain that

$$\sigma_{X_2 - X_1}^2 = o\left(\frac{\nu_c^{\frac{2}{n}}}{M^{\frac{4}{n}}}\right), \text{ leading to } \rho = 1 - o\left(\frac{\nu_c^{\frac{2}{n}}}{M^{\frac{4}{n}}}\right). \quad (32)$$

This implies that r_0 can be chosen such that $r_0 = o(\nu_c^{\frac{1}{n}})$, while (28) still holds.

On the other hand, for certain distributions Markov's inequality yields a loose bound, rendering the sufficient condition (32) too restrictive. This may happen if the distribution of

the random variable $\|X_2^n - X_1^n\|^2$ has a light tail or a bounded support. For instance, if the random variable $|X_2 - X_1|$ has as support the interval $[0, \frac{r_0}{\sqrt{n}}]$ for r_0 such that $\frac{r_0}{r_c}$ is constant under (22), then we can have $\sigma_{X_2 - X_1}^2 = \Theta(\nu_c^{\frac{2}{n}})$, while $\mathcal{P}_{X_1 X_2}(r_0) = 0$.

Next we will address the situation when (32) holds. In this case, according to Theorem 2 and Corollary 1, we have

$$\begin{aligned} d_{s,i}(\mathcal{L}^{(n,r_0)}) &= \frac{1}{4} G_{s/2} M^{\frac{4}{n}} K^{\frac{2}{n}} \nu_c^{\frac{2}{n}} (1 + o(1)), \\ d_{c,i}(\mathcal{L}^{(n,r_0)}) &= G_c \nu_c^{\frac{2}{n}} (1 + o(1)). \end{aligned}$$

Then relation (32) is equivalent to $(1 - \rho) \frac{d_{s,i}(\mathcal{L}^{(n,r_0)})}{d_{c,i}(\mathcal{L}^{(n,r_0)})^2} \rightarrow 0$, and further the limits in (22) are equivalent to

$$\begin{aligned} d_{s,i}(\mathcal{L}^{(n,r_0)}) &\rightarrow 0, \quad \frac{d_{c,i}(\mathcal{L}^{(n,r_0)})}{d_{s,i}(\mathcal{L}^{(n,r_0)})} \rightarrow 0, \\ (1 - \rho) \frac{d_{s,i}(\mathcal{L}^{(n,r_0)})}{d_{c,i}(\mathcal{L}^{(n,r_0)})^2} &\rightarrow 0. \end{aligned} \quad (33)$$

Let us make the notations

$$d_s = \frac{\sum_{i=1}^2 d_{s,i}(\mathcal{L}^{(n,r_0)})}{2}, \quad d_c = \frac{\sum_{i=1}^2 d_{c,i}(\mathcal{L}^{(n,r_0)})}{2}. \quad (34)$$

Then the limits in (33) imply that

$$d_s \rightarrow 0, \quad \frac{d_c}{d_s} \rightarrow 0, \quad (1 - \rho) \frac{d_s}{d_c^2} \rightarrow 0. \quad (35)$$

Let us denote by $R_L(n, d_s, d_c)$ ¹ the infimum of $R(\mathcal{L}^{(n,r_0)})$ over all $\mathcal{L}^{(n,r_0)}$ satisfying (34) for fixed n and fixed pair (X_1, X_2) . Assume that the lattices used in the construction achieve the smallest second moment for the corresponding dimension, denoted by $G_{opt,n}$. Applying this result in Corollary 1, we further obtain that

$$\left| R_L(n, d_s, d_c) - 2h(X) - \frac{1}{2} \log_2 \frac{G_{opt,n}^2}{4d_s d_c} \right| \leq \zeta(n, d_s, d_c, \rho), \quad (36)$$

where $\lim_{(35)} \zeta(n, d_s, d_c, \rho) = 0$.

Let us turn our attention to the case when $X_1 = X_2 = X$, i.e., $\rho = 1$. In this case the asymptotic regime (35) is specified only by $d_s \rightarrow 0$ and $\frac{d_c}{d_s} \rightarrow 0$. We will show that in this case our scheme achieves the fundamental limit of MDC. In order to formalize the result, we define the operational rate-distortion function of the proposed LRDCS in the case when $\rho = 1$ as follows

$$R_L(d_s, d_c) \triangleq \inf_{n \geq 1} R_L(n, d_s, d_c).$$

Theorem 3. For any source X with continuous pdf, the following holds

$$\lim_{\substack{d_s \rightarrow 0 \\ \frac{d_c}{d_s} \rightarrow 0}} (R_L(d_s, d_c) - R_{MD}(d_s, d_c)) = 0,$$

where $R_{MD}(d_s, d_c)$ was defined in Section III.

¹This quantity is defined for those triples (n, d_s, d_c) for which there exists an LRDCS $\mathcal{L}^{(n,r_0)}$ achieving average side distortion d_s and average central distortion d_c .

Proof: Applying relation (36) in the case of $\rho = 1$ leads to

$$\lim_{\substack{d_s \rightarrow 0 \\ \frac{d_c}{d_s} \rightarrow 0}} \left(R_L(n, d_s, d_c) - 2h(X) - \frac{1}{2} \log_2 \frac{G_{opt,n}^2}{4d_s d_c} \right) = 0. \quad (37)$$

Using the fact that $\lim_{n \rightarrow \infty} G_{opt,n} = \frac{1}{2\pi e}$ [43] together with relation (8) further leads to

$$\lim_{n \rightarrow \infty} \lim_{\substack{d_s \rightarrow 0 \\ \frac{d_c}{d_s} \rightarrow 0}} (R_L(n, d_s, d_c) - R_{MD}(d_s, d_c)) = 0. \quad (38)$$

The above relation implies that for every $\epsilon > 0$, there are $n(\epsilon)$ and $\delta(\epsilon, n)$, for $n \geq n(\epsilon)$, such that $R_L(n, d_s, d_c) - R_{MD}(d_s, d_c) < \epsilon$, for all $n \geq n(\epsilon)$ and $d_s \leq \delta(\epsilon, n)$, $d_c/d_s \leq \delta(\epsilon, n)$. Let $\xi(\epsilon) = \delta(\epsilon, n(\epsilon))$. Then whenever $d_s \leq \xi(\epsilon)$ and $d_c/d_s \leq \xi(\epsilon)$, we have $R_L(d_s, d_c) - R_{MD}(d_s, d_c) \leq R_L(n(\epsilon), d_s, d_c) - R_{MD}(d_s, d_c) < \epsilon$. This implies that

$$\lim_{\substack{d_s \rightarrow 0 \\ \frac{d_c}{d_s} \rightarrow 0}} (R_L(d_s, d_c) - R_{MD}(d_s, d_c)) \leq 0.$$

Since the inequality $R_L(d_s, d_c) - R_{MD}(d_s, d_c) \geq 0$ holds for all pairs (d_s, d_c) , the claim of the theorem follows. ■

Note that another RDSC scheme which achieves the fundamental limit of MDC is a scheme which uses the encoders and decoders of an MDLVQ. Therefore, it is interesting to find out whether there is any advantage in using the proposed LRDSC scheme rather than directly applying an MDLVQ.

More specifically, in an MDLVQ-based RDSC system, encoder i maps the input sequence x_i^n to $\lambda_{c,i} = Q_c(x_i^n)$, next applies the index assignment $\alpha = (\alpha_1, \alpha_2) : \Lambda_c \rightarrow \Lambda_s \times \Lambda_s$ and outputs the side lattice point $\alpha_i(\lambda_{c,i})$. Side decoder i uses the received side lattice point $\lambda_{s,i}$ as the source reconstruction, while the central decoder looks for the central lattice point λ_c satisfying $(\lambda_{s,1}, \lambda_{s,2}) = (\alpha_1(\lambda_c), \alpha_2(\lambda_c))$, and uses λ_c as the common reconstruction for both sources. The problem with this scheme is that, when $\lambda_{c,1} \neq \lambda_{c,2}$, the central distortion is essentially as high as the side distortion. To see this, note first that the mappings α_1, α_2 are constructed such that $\alpha_1(\lambda'_c) + \alpha_2(\lambda'_c) = 2Q_{s/2}(\lambda'_c)$ for each $\lambda'_c \in \Lambda_c$. Assume now that $Q_{s/2}(\lambda_{c,1}) = Q_{s/2}(\lambda_{c,2}) = \tau$ and $\lambda_{c,1} \neq \lambda_{c,2}$. Then $\alpha_1(\lambda_{c,1}) \neq \alpha_1(\lambda_{c,2})$ because otherwise we would also have $\alpha_2(\lambda_{c,1}) = \alpha_2(\lambda_{c,2})$, contradicting the fact that α is injective. Further, we obtain that $\alpha_1(\lambda_{c,1}) + \alpha_2(\lambda_{c,2}) \neq 2\tau$, which implies that the point λ_c chosen by the central decoder is not in the same Voronoi region of the fractional lattice $\Lambda_{s/2}$ as $\lambda_{c,1}$ and $\lambda_{c,2}$. Then if $\|\lambda_{c,i} - \tau\| < 1/2r_{s/2}$, the error in the reconstruction is at least $1/2r_{s/2}$. If $\sigma_{X_2 - X_1}^2 = \Theta(r_0^2)$ and $\frac{r_0}{r_c}$ is constant as the limits in (22) are approached, the probability that $Q_{s/2}(\lambda_{c,1}) = Q_{s/2}(\lambda_{c,2}) = \tau$ and $\lambda_{c,1} \neq \lambda_{c,2}$ does not approach 0, thus the central distortion cannot satisfy relation (29), while the proposed RDSC scheme can.

VI. DETAILED OPERATION OF THE PROPOSED LRDSC SCHEME

This section presents in detail the operation of the proposed LRDSC.

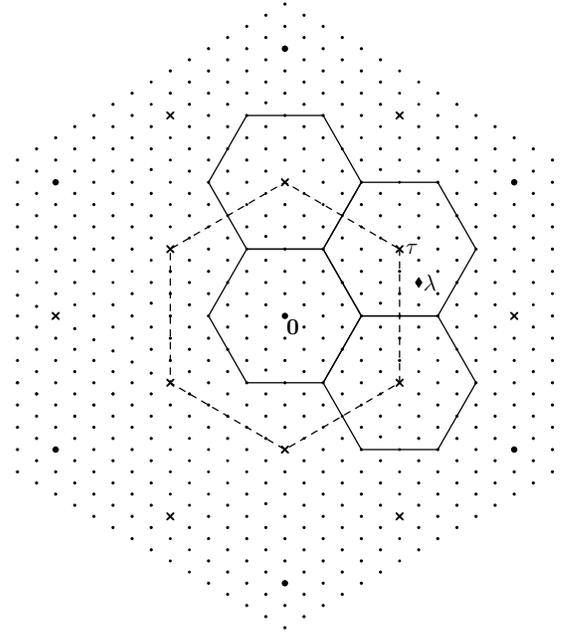


Fig. 2. Illustration of the lattices involved in the proposed scheme. The big dots represent points in Λ_s , while the small dots correspond to points in Λ_{in} . The value of μ is 12. The crosses represent the points in $\Lambda_{s/2}$ which are not in Λ_s . The hexagon drawn with dashed lines is the boundary of $V_s(\mathbf{0})$. The centers of the four hexagons drawn with solid lines form the set \mathcal{T} . Each such hexagon is the boundary of a Voronoi region with respect to the lattice $\Lambda_{s/2}$. The point λ represented by a diamond is an example of a point in the set \mathcal{U} , while $\tau = Q_{s/2}(\lambda)$.

A. Preliminaries

First we prove a key property, enabled by condition (20), which is essential in the operation of the LRDSC.

Lemma 1. *If $x_2^n - x_1^n \in \mathcal{B}_{r_0}$, then*

$$\begin{aligned} \|Q_c(x_1^n) - Q_c(x_2^n)\| &< r_{in}, \\ \|Q_{in}(Q_c(x_1^n)) - Q_{in}(Q_c(x_2^n))\| &< 3\bar{r}_{in}. \end{aligned}$$

Proof: Let $\lambda_{c,i} \triangleq Q_c(x_i^n)$ and $\lambda_i \triangleq Q_{in}(\lambda_{c,i})$, for $i = 1, 2$. Using the triangle inequality repeatedly, followed by (20), one obtains that

$$\begin{aligned} \|\lambda_{c,1} - \lambda_{c,2}\| &\leq \|\lambda_{c,1} - x_1^n\| + \|x_1^n - x_2^n\| + \|x_2^n - \lambda_{c,2}\| \\ &< r_0 + 2\bar{r}_c \leq r_{in}. \end{aligned}$$

Additionally,

$$\begin{aligned} \|\lambda_1 - \lambda_2\| &\leq \|\lambda_1 - \lambda_{c,1}\| + \|\lambda_{c,1} - \lambda_{c,2}\| + \|\lambda_{c,2} - \lambda_2\| \\ &< r_{in} + 2\bar{r}_{in} < 3\bar{r}_{in}, \end{aligned}$$

which completes the proof. ■

Next we define the labeling function $\beta_i : \Lambda_{in} \rightarrow \Lambda_s$ used at encoder $i = 1, 2$. For this we need to introduce some more notations as follows. Let $\mathcal{T} \triangleq V_s(\mathbf{0}) \cap \Lambda_{s/2}$. Then \mathcal{T} is a set of coset representatives of Λ_s relative to $\Lambda_{s/2}$. Thus, we have $|\mathcal{T}| = N(\Lambda_s : \Lambda_{s/2}) = 2^n$ and

$$\Lambda_{s/2} = \bigcup_{\tau \in \mathcal{T}} (\tau + \Lambda_s).$$

It can be easily seen that the set $\cup_{\tau \in \mathcal{T}} V_{s/2}(\tau)$ is a fundamental cell of Λ_s . Denote $\mathcal{U} \triangleq \cup_{\tau \in \mathcal{T}} V_{s/2}(\tau) \cap \Lambda_{in}$. Then \mathcal{U} is a set

of coset representatives of Λ_s relative to Λ_{in} , which implies that $|\mathcal{U}| = N(\Lambda_s : \Lambda_{in}) = M$ and

$$\Lambda_{in} = \bigcup_{\lambda \in \mathcal{U}} (\lambda + \Lambda_s).$$

We will first define β_i for $\lambda \in \mathcal{U}$ as follows

$$\beta_1(\lambda) \triangleq \mu(\lambda - \tau), \quad \beta_2(\lambda) \triangleq 2\tau - \mu(\lambda - \tau), \quad (39)$$

where $\tau = Q_{s/2}(\lambda)$. Further, the mappings β_1 and β_2 are extended to Λ_{in} using shifting. For arbitrary $\lambda \in \Lambda_{in}$, let $\lambda_{s/2} = Q_{s/2}(\lambda)$, i.e., $\lambda \in V_{s/2}(\lambda_{s/2})$. Then there is a unique pair $(\tau, \lambda_s) \in \mathcal{T} \times \Lambda_s$ such that $\lambda_{s/2} = \lambda_s + \tau$. More specifically, we have $\lambda_s = Q_s(\lambda_{s/2})$ and $\tau = \lambda_{s/2} \bmod \Lambda_s$. Then we define

$$\begin{aligned} \beta_1(\lambda) &\triangleq \beta_1(\lambda - \lambda_s) + \lambda_s = \mu(\lambda - \lambda_s - \tau) + \lambda_s, \\ \beta_2(\lambda) &\triangleq \beta_2(\lambda - \lambda_s) + \lambda_s = 2\tau - \mu(\lambda - \lambda_s - \tau) + \lambda_s. \end{aligned}$$

The above definition implies that the mappings β_i satisfy the *shift-invariance property*, i.e., that

$$\beta_i(\lambda + \lambda'_s) = \beta_i(\lambda) + \lambda'_s, \quad \forall \lambda \in \Lambda_{in}, \quad \forall \lambda'_s \in \Lambda_s, \quad i = 1, 2.$$

The shift-invariance property further leads to the following relations, for $i = 1, 2$,

$$\beta_i^{-1}(\lambda_s) = \beta_i^{-1}(\mathbf{0}) + \lambda_s, \quad \forall \lambda_s \in \Lambda_s, \quad (40)$$

$$\beta_i^{-1}(\mathbf{0}) = \{\lambda - \beta_i(\lambda) | \lambda \in \mathcal{U}\}. \quad (41)$$

Relation (40) is obvious. In order to prove (41), consider $\lambda' \in \Lambda_{in}$ and let $(\lambda, \lambda_s) \in \mathcal{U} \times \Lambda_s$ be the unique pair such that $\lambda' = \lambda + \lambda_s$. The shift-invariance property implies that $\beta_i(\lambda') = \beta_i(\lambda) + \lambda_s$, which leads to $\lambda_s = \beta_i(\lambda') - \beta_i(\lambda)$. Further, we obtain that $\lambda' = \lambda + \beta_i(\lambda') - \beta_i(\lambda)$. Consequently, the equality $\beta_i(\lambda') = 0$ is equivalent to $\lambda' = \lambda - \beta_i(\lambda)$, which proves the claim.

We point out that the construction of the mappings β_1 and β_2 was inspired by the index assignment used in MDLVQ [30], [31] in two ways: 1) by defining the mappings on a set of coset representatives first and then extending them by shifting; 2) by imposing the condition that $\beta_1(\lambda) + \beta_2(\lambda) = 2Q_{s/2}(\lambda)$ for each $\lambda \in \Lambda_{in}$. On the other hand, it is important to note that we cannot simply use the mappings $\alpha_1, \alpha_2 : \Lambda_{in} \rightarrow \Lambda_s$ that define the index assignment for MDLVQ² in [30], [31] in place of our mappings β_1, β_2 , since the requirement at the central decoder in our case is stronger than for MDLVQ. In particular, based on a received pair of side lattice points $\lambda_{s,1}, \lambda_{s,2}$, the central decoder of the MDLVQ uniquely identifies a point $\lambda \in \Lambda_{in}$ such that $(\alpha_1(\lambda), \alpha_2(\lambda)) = (\lambda_{s,1}, \lambda_{s,2})$. However, as we will see shortly, the central decoder in our scheme needs to uniquely identify two points $\lambda_1, \lambda_2 \in \Lambda_{in}$ such that $(\beta_1(\lambda_1), \beta_2(\lambda_2)) = (\lambda_{s,1}, \lambda_{s,2})$, using the additional knowledge of $\lambda_1 - \lambda_2$. Using the pair of mappings (α_1, α_2) designed for the MDLVQ in place of (β_1, β_2) does not guarantee that the latter requirement is satisfied.

²The lattice Λ_{in} takes here the place of the central lattice on which the index assignment is defined for MDLVQ.

B. LRDSC Operation

Before describing the details of the proposed scheme we need the following discussion. Let us denote $\lambda_i = Q_{in}(Q_c(x_i^n))$, $i = 1, 2$. Our scheme is designed such that side decoder i will always be able to recover $\beta_i(\lambda_i)$, while the central decoder recovers $\lambda_{c,i} = Q_c(x_i^n)$, $i = 1, 2$, when the input sequences are sufficiently close, i.e., when $x_2^n - x_1^n \in \mathcal{B}_{r_0}$. However, for the central decoder to achieve this goal, some additional information needs to be transmitted besides $\beta_1(\lambda_1)$ and $\beta_2(\lambda_2)$. The amount of this additional information is smaller when λ_1 and λ_2 are both in the same Voronoi cell of the lattice $\Lambda_{s/2}$. Encoder i is not able to determine all the time if this is the case or not, since it does not have knowledge of the other source sequence. However, based on Lemma 1, if $\lambda_i \in V_{s/2}(\lambda_{s/2})$ and the distance from λ_i to the boundary of $V_{s/2}(\lambda_{s/2})$ is not smaller than $3\bar{r}_{in}$, then encoder i can infer that the other sequence is also in $V_{s/2}(\lambda_{s/2})$ when $x_2^n - x_1^n \in \mathcal{B}_{r_0}$. Thus, we define the set

$$\mathcal{C} \triangleq \bigcup_{\lambda_{s/2} \in \Lambda_{s/2}} \mathcal{C}(\lambda_{s/2}), \quad (42)$$

where

$$\mathcal{C}(\lambda_{s/2}) \triangleq V_{s/2}(\lambda_{s/2}) \setminus (\lambda_{s/2} + \gamma V_{s/2}(\mathbf{0})),$$

for $\gamma \triangleq 1 - \frac{3\bar{r}_{in}}{r_{s/2}}$. According to Lemma 1, if $\lambda_i \notin \mathcal{C}$, then λ_{3-i} is in the same Voronoi cell of $\Lambda_{s/2}$ as λ_i , when $x_2^n - x_1^n \in \mathcal{B}_{r_0}$. Now we are ready to present the details of the encoder and decoder operation.

Encoder. Encoder i , for $i = 1, 2$, operates as follows (Fig. 3). First the input sequence x_i^n is quantized to the closest central lattice point $\lambda_{c,i} \triangleq Q_c(x_i^n)$. Next the point $\lambda_{c,i}$ is quantized to the closest point in the lattice Λ_{in} , $\lambda_i \triangleq Q_{in}(\lambda_{c,i})$. Let $u_i \triangleq \lambda_{c,i} \bmod \Lambda_{in}$ and $\lambda_{s,i} \triangleq \beta_i(\lambda_i)$. Then encoder i outputs $\lambda_{s,i}$, u_i and b_i , where $b_i = 1$ if $\lambda_i \in \mathcal{C}$ and $b_i = 0$ otherwise. Moreover, if $b_i = 1$, encoder i also transmits $\tau_i \triangleq Q_{s/2}(\lambda_i) \bmod \Lambda_s$. The first component, $\lambda_{s,i}$, will be used at the side decoder i , therefore, it is compressed using entropy coding before transmission. On the other hand, u_1 and u_2 are used only at the central decoder, therefore they will be compressed using Slepian-Wolf coding. Finally, b_i and τ_i will also be used only at the central decoder, thus they may be compressed using Slepian-Wolf coding. However, we will use entropy coding to encode b_i and fixed length codes for τ_i for simplicity of analysis, since, as shown in the proof of Theorem 2, the rate overhead is negligible asymptotically. Note that the aforementioned entropy coders and Slepian-Wolf coders are applied to blocks of m symbols, where m approaches ∞ .

Decoder. Side decoder i , for $i = 1, 2$, outputs the reconstruction $\hat{x}_{s,i}^n \triangleq \lambda_{s,i}$. The central decoder recovers both values $\lambda_{s,1}$ and $\lambda_{s,2}$, and additionally, u_1, u_2, b_1, b_2 . First the decoder checks if the following condition is satisfied

$$\|\lambda_{s,1} - \lambda_{s,2}\| \leq (8 + c)\bar{r}_s + 3\bar{r}_{in}. \quad (43)$$

If the condition is violated, then the decoder concludes that $x_2^n - x_1^n \notin \mathcal{B}_{r_0}$, and outputs $\lambda_{s,i}$ as the reconstruction for source i , i.e., $\hat{x}_{c,i}^n \triangleq \lambda_{s,i}$, for $i = 1, 2$.

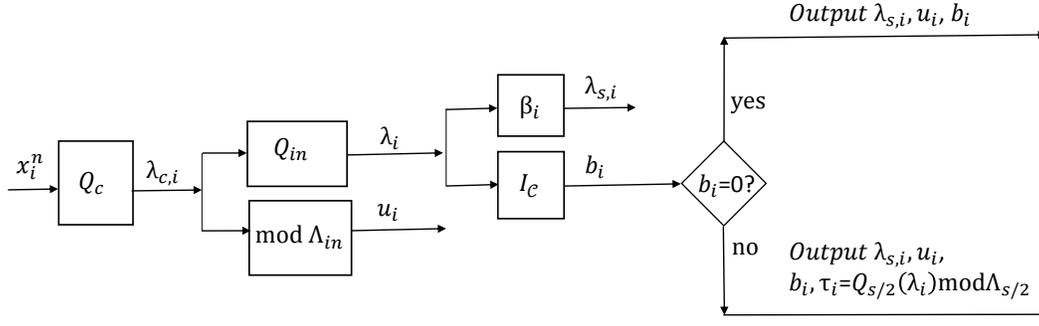


Fig. 3. Diagram describing the operation of encoder i , for $i = 1, 2$. I_C denotes the indicator function of the set C .

If condition (43) is satisfied, the decoder assumes that $x_2^n - x_1^n \in \mathcal{B}_{r_0}$ and proceeds as follows. First the following is computed

$$\tilde{\lambda} \triangleq Q_{in}(u_1 - u_2). \quad (44)$$

Next the decoder proceeds based on the values of b_1 and b_2 , and of τ_1 and τ_2 (if applicable), according to the following cases.

- 1) If $b_1 = 0$ or $b_2 = 0$, the decoder evaluates

$$\tilde{\lambda}_{s/2} \triangleq 1/2(\lambda_{s,1} + \lambda_{s,2} + \mu\tilde{\lambda}), \quad (45)$$

$$\tilde{\tau} \triangleq \tilde{\lambda}_{s/2} \bmod \Lambda_s, \quad (46)$$

$$\tilde{\lambda}_1 \triangleq \tilde{\lambda}_{s/2} + \frac{1}{\mu}(\lambda_{s,1} - \tilde{\lambda}_{s/2} + \tilde{\tau}), \quad (47)$$

$$\tilde{\lambda}_2 \triangleq \tilde{\lambda}_{s/2} + \frac{1}{\mu}(\tilde{\tau} + \tilde{\lambda}_{s/2} - \lambda_{s,2}), \quad (48)$$

and outputs the reconstructions $\hat{x}_{c,i}^n \triangleq \tilde{\lambda}_i + u_i$, for $i = 1, 2$.

- 2) If $b_1 = b_2 = 1$ and $\tau_1 = \tau_2$, the decoder proceeds as in case 1).
3) If $b_1 = b_2 = 1$ and $\tau_1 \neq \tau_2$, then the decoder computes

$$\tilde{v} \triangleq 1/2(\lambda_{s,1} + \lambda_{s,2} + \mu\tilde{\lambda} - 2\tau_2 - \mu(\tau_2 - \tau_1)), \quad (49)$$

$$\hat{w} \triangleq \tilde{v} \bmod \Lambda_s, \quad (50)$$

$$\tilde{w} \triangleq \hat{w} - Q_s(\hat{w} + \frac{1}{2}(\tau_2 - \tau_1)), \quad (51)$$

$$\tilde{\lambda}_s \triangleq \tilde{v} - (\mu + 1)\tilde{w}, \quad \tilde{\lambda}'_s \triangleq \tilde{\lambda}_s + 2\tilde{w}, \quad (52)$$

$$\tilde{\lambda}_1 \triangleq \tilde{\lambda}_s + \tau_1 + \frac{1}{\mu}(\lambda_{s,1} - \tilde{\lambda}_s), \quad (53)$$

$$\tilde{\lambda}_2 \triangleq \tilde{\lambda}'_s + \tau_2 + \frac{1}{\mu}(2\tau_2 + \tilde{\lambda}'_s - \lambda_{s,2}). \quad (54)$$

Finally, the reconstructions are computed as $\hat{x}_{c,i}^n \triangleq \tilde{\lambda}_i + u_i$, for $i = 1, 2$.

Proposition 1. Let $\lambda_{c,i} \triangleq Q_c(x_i^n)$, $\lambda_i \triangleq Q_{in}(\lambda_{c,i})$, $u_i \triangleq \lambda_{c,i} \bmod \Lambda_{in}$, $\lambda_{s,i} \triangleq \beta_i(\lambda_i)$ and $\tau_i \triangleq Q_{s/2}(\lambda_i) \bmod \Lambda_s$, for $i = 1, 2$. Then when $x_2^n - x_1^n \in \mathcal{B}_{r_0}$ and the Slepian-Wolf decoding of u_1 and u_2 is successful, we have $\hat{x}_{c,i}^n = \lambda_{c,i}$, for $i = 1, 2$, and μ sufficiently large.

It is worth pointing out that a crucial aspect of the proposed scheme is the use of an identical quantizer for both sources

(the quantizer defined by the central lattice). The use of a common finite-length quantizer in the two-source distributed coding scenario was advocated earlier by Shirani and Pradhan in [44] who argue that such a design preserves the correlation between sources more efficiently.

VII. CONCLUSION

We have proposed a constructive lattice-based scheme for robust distributed coding of two correlated sources. The analysis shows, among other things, that, in the asymptotic regime where 1) the side distortion approaches 0 and 2) the ratio between the central and side distortions approaches 0, our scheme is capable of approaching the information-theoretic limit of quadratic MDC when the two sources are identical, whereas a variant of the random-coding-based RDSC scheme by Chen and Berger with Gaussian codes is strictly sub-optimal. Note that in standard random coding arguments, to facilitate the joint typicality analysis, the block-length is often sent to ∞ . However, in the infinite block-length limit, the condition needed to ensure joint typicality in the distributed setting is much more restrictive than its counterpart in the centralized setting; as a consequence, the resulting distributed coding schemes, when specialized to the centralized setting, may fail to achieve the fundamental performance limit. In contrast, for lattice-based schemes, the performance analysis can be carried out under fixed block-length (i.e., fixed lattice dimension), which reveals a smooth transition from the distributed setting to the centralized setting. In this sense, our result echoes the recent finding in [44] regarding the importance of finite block-length schemes in distributed source coding.

APPENDIX A

PROOF OF PROPOSITION 1

Proof of Proposition 1: Assume that $x_2^n - x_1^n \in \mathcal{B}_{r_0}$ and that the Slepian-Wolf decoder employed at the central decoder is able to recover u_1 and u_2 correctly. First we need to prove that condition (43) is satisfied. To this end, we first show that

$$\bar{r}(\beta^{-1}(\mathbf{0})) \leq (4 + \mu/2)\bar{r}_s. \quad (55)$$

Note that relation (41) leads to

$$\bar{r}(\beta_i^{-1}(\mathbf{0})) \leq \bar{r}(\mathcal{U}) + \bar{r}(\beta_i(\mathcal{U})). \quad (56)$$

Further, since $\mathcal{T} \subset V_s(\mathbf{0})$ and $V_{s/2}(\mathbf{0}) \subset V_s(\mathbf{0})$, we obtain that $\mathcal{U} \subset \cup_{\tau \in \mathcal{T}} (\tau + V_{s/2}(\mathbf{0})) \subset 2V_s(\mathbf{0})$. Thus, $\bar{r}(\mathcal{U}) \leq 2\bar{r}_s$. Moreover, from the definition of β_i given in (39), we obtain that $\bar{r}(\beta_i(\mathcal{U})) \leq 2\bar{r}(\mathcal{T}) + \mu\bar{r}_{s/2} \leq 2\bar{r}_s + \mu\bar{r}_{s/2}$. The above discussion, together with relation (56) and the fact that $\bar{r}_{s/2} = 1/2\bar{r}_s$, implies (55).

By applying the triangle inequality and the fact that $\|\lambda - \beta_i(\lambda)\| \leq \bar{r}(\beta^{-1}(\mathbf{0}))$, together with Lemma 1, we obtain

$$\begin{aligned} \|\lambda_{s,1} - \lambda_{s,2}\| &\leq \|\lambda_{s,1} - \lambda_1\| + \|\lambda_1 - \lambda_2\| + \|\lambda_2 - \lambda_{s,2}\| \\ &\leq 2\bar{r}(\beta^{-1}(\mathbf{0})) + 3\bar{r}_{in}. \end{aligned}$$

Combining the above with (55) proves relation (43).

Using Lemma 1 and the fact that $\lambda_{c,i} = \lambda_i + u_i$, $i = 1, 2$, we obtain that

$$r_{in} > \|\lambda_{c,1} - \lambda_{c,2}\| = \|u_1 - u_2 - (\lambda_2 - \lambda_1)\|,$$

which, together with the fact that $\lambda_2 - \lambda_1 \in \Lambda_{in}$, implies that $u_1 - u_2 \in V_{in}(\lambda_2 - \lambda_1)$, i.e., $\lambda_2 - \lambda_1 = Q_{in}(u_1 - u_2)$. This further implies that $\tilde{\lambda}$ computed in (44) satisfies the equality

$$\tilde{\lambda} = \lambda_2 - \lambda_1. \quad (57)$$

Let $\lambda_s \triangleq Q_s(Q_{s/2}(\lambda_1))$ and $\lambda'_s \triangleq Q_s(Q_{s/2}(\lambda_2))$. Using the fact that $\tau_i \triangleq Q_{s/2}(\lambda_i) \bmod \Lambda_s$, for $i = 1, 2$, it follows that $\lambda_1 \in V_{s/2}(\lambda_s + \tau_1)$ and $\lambda_2 \in V_{s/2}(\lambda'_s + \tau_2)$. Moreover, since $\lambda_{s,i} = \beta_i(\lambda_i)$ for $i = 1, 2$, we obtain that

$$\lambda_{s,1} = \mu(\lambda_1 - \lambda_s - \tau_1) + \lambda_s, \quad (58)$$

$$\lambda_{s,2} = \lambda'_s + 2\tau_2 - \mu(\lambda_2 - \lambda'_s - \tau_2). \quad (59)$$

Assume now that case 1) holds. According to Lemma 1, we have $\lambda_s + \tau_1 = \lambda'_s + \tau_2$. Since $\tau_1, \tau_2 \in \mathcal{T}$, it follows that $\lambda_s = \lambda'_s$ and $\tau_1 = \tau_2$. Using further equations (45), (46), (57), (58) and (59), we obtain that $\tilde{\lambda}_{s/2} = \lambda_s + \tau_1$. This implies that $\tau_1 = \tilde{\lambda}_{s/2} \bmod \Lambda_s$, i.e., $\tilde{\tau} = \tau_1$. Equations (47) and (48) imply that $\lambda_i = \lambda_i$ and further that $\hat{x}_{c,i}^n = \lambda_{c,i}$, for $i = 1, 2$.

Assume now that $b_1 = b_2 = 1$. Let $\Delta_{s/2}$ denote the smallest distance between two points belonging, respectively, to the closures of two non-adjacent Voronoi regions of lattice $\Lambda_{s/2}$. When μ is large enough,

$$\Delta_{s/2} > 3\bar{r}_{in}. \quad (60)$$

Recall that, according to Lemma 1, we have $\|\lambda_1 - \lambda_2\| < 3\bar{r}_{in}$. Condition (60) further ensures that $\|\lambda_1 - \lambda_2\| < \Delta_{s/2}$, which implies that $V_{s/2}(\lambda_s + \tau_1)$ and $V_{s/2}(\lambda'_s + \tau_2)$ are either identical or adjacent. Further, if $\tau_1 = \tau_2$, it follows that $\lambda'_s + \tau_2 - (\lambda_s + \tau_1) \in \Lambda_s$. Thus, $V_{s/2}(\lambda_s + \tau_1)$ and $V_{s/2}(\lambda'_s + \tau_2)$ cannot be adjacent. Consequently, the equality $\lambda'_s + \tau_2 = \lambda_s + \tau_1$ holds and the proof proceeds as in case 1).

Assume now that $\tau_1 \neq \tau_2$. Then $\lambda_s + \tau_1 \neq \lambda'_s + \tau_2$. Denote $\Delta\lambda_{s/2} \triangleq \lambda'_s + \tau_2 - (\lambda_s + \tau_1)$. Then $\mathbf{0}$ and $\Delta\lambda_{s/2}$ are adjacent points of the lattice $\Lambda_{s/2}$ (i.e., their Voronoi regions are adjacent). It follows that

$$\Delta\lambda_{s/2} \in \overline{V_s(\mathbf{0})}. \quad (61)$$

Let $w \triangleq \frac{1}{2}(\lambda'_s - \lambda_s)$. Using equations (49), (50), (57), (58) and (59), we obtain that

$$\tilde{w} = \lambda_s + \frac{\mu}{2}(\lambda'_s - \lambda_s) + w. \quad (62)$$

Since μ is even, it follows that $\frac{\mu}{2}(\lambda'_s - \lambda_s) \in \Lambda_s$. Thus, $w \bmod \Lambda_s = \tilde{w} \bmod \Lambda_s = \hat{w}$. It follows that $w = \tilde{\lambda}_s + \hat{w}$ for some $\tilde{\lambda}_s \in \Lambda_s$. Then $\Delta\lambda_{s/2} = 2w + \tau_2 - \tau_1 = 2(\tilde{\lambda}_s + \hat{w}) + \tau_2 - \tau_1$. Using further (61) leads to $\frac{1}{2}\Delta\lambda_{s/2} = \tilde{\lambda}_s + \hat{w} + \frac{1}{2}(\tau_2 - \tau_1) \in \frac{1}{2}\overline{V_s(\mathbf{0})} \subset V_s(\mathbf{0})$, which further implies that $-\tilde{\lambda}_s = Q_s(\hat{w} + \frac{1}{2}(\tau_2 - \tau_1))$. It follows that $\tilde{w} = w$, where \tilde{w} is defined in (51). Combining this with (52) and (62), we obtain that $\tilde{\lambda}_s = \lambda_s$ and $\tilde{\lambda}'_s = \lambda'_s$. Finally, equations (53) and (54) imply that $\lambda_i = \lambda_i$ and further that $\hat{x}_{c,i}^n = \lambda_{c,i}$, for $i = 1, 2$. ■

APPENDIX B

PROOF OF RESULTS IN SECTION V

Before proceeding to the proof of Theorem 2, we need a few more notations and some auxiliary results.

Consider an LRDCS $\mathcal{L}^{(n,r_0)} = (\Lambda_s, \Lambda_{in}, \Lambda_c)$. For each $\lambda_s \in \Lambda_s$ and $i = 1, 2$, let $\mathcal{A}_i(\lambda_s) \triangleq \{x_i^n | \hat{x}_{s,i}^n = \lambda_s\}$. Further, for each $\lambda \in \Lambda_{in}$, denote $\mathcal{M}(\lambda) \triangleq \cup_{\lambda_c \in V_{in}(\lambda) \cap \Lambda_c} V_c(\lambda_c)$. Then $\mathcal{A}_i(\lambda_s) = \cup_{\lambda \in \beta_i^{-1}(\lambda_s)} \mathcal{M}(\lambda)$. Clearly, we have $\mathcal{M}(\lambda) = \lambda + \mathcal{M}(\mathbf{0})$ for all $\lambda \in \Lambda$. This fact, together with relation (40), implies that

$$\mathcal{A}_i(\lambda_s) = \mathcal{A}_i(\mathbf{0}) + \lambda_s, \quad \forall \lambda_s \in \Lambda_s. \quad (63)$$

Obviously, we have $d_{s,i}(\mathcal{L}^{(n,r_0)}) = D(Q_{\mathcal{A}_i}, X_i^n)$, where $Q_{\mathcal{A}_i}$ denotes the quantizer which maps each input sequence $x_i^n \in \mathcal{A}_i(\lambda_s)$ to λ_s , for $\lambda_s \in \Lambda_s$. Since $\|x_i^n - \hat{x}_{c,i}^n\| \geq \|x_i^n - Q_c(x_i^n)\|$, it follows that

$$d_{c,i}(\mathcal{L}^{(n,r_0)}) \geq D(Q_c, X_i^n).$$

Further, let us denote $\Delta_{i,sup}(\mathcal{L}^{(n,r_0)}) \triangleq \sup_{x_i^n \in \mathbb{R}^n} \|x_i^n - \hat{x}_{c,i}^n\|$, $i = 1, 2$. Additionally, let $\mathcal{P}_{e,SW}$ denote the probability that the Slepian-Wolf decoder fails. In view of the definition of $\mathcal{P}_{X_1 X_2}(r_0)$ (see (21)) and Proposition 1, it follows that, for $i = 1, 2$,

$$d_{c,i}(\mathcal{L}^{(n,r_0)}) \leq \frac{1}{n}(\mathcal{P}_{X_1 X_2}(r_0) + \mathcal{P}_{e,SW})\Delta_{i,sup}^2 + D(Q_c, X_i^n).$$

The following lemma, proved in Appendix D, gives an upper bound for $\Delta_{i,sup}$.

Lemma 2. *There is some constant κ_0 such that, for each $i = 1, 2$, each positive integer n , and each LRDCS $\mathcal{L}^{(n,r_0)}$,*

$$\Delta_{i,sup}(\mathcal{L}^{(n,r_0)}) \leq \kappa_0 (M\nu_s)^{\frac{1}{n}}.$$

It is known that the probability that the Slepian-Wolf decoder fails can be made arbitrarily small by increasing the block length used for Slepian-Wolf encoding. Since $\Delta_{i,sup}(\mathcal{L}^{(n,r_0)})$ is bounded, it follows that the impact on the distortion of the Slepian-Wolf decoder failure can also be made arbitrarily small. Therefore, in the limit as the block length of the Slepian-Wolf encoder approaches infinity,

$$\begin{aligned} D(Q_c, X_i^n) &\leq d_{c,i}(\mathcal{L}^{(n,r_0)}) \leq \frac{1}{n}\kappa_0^2 \mathcal{P}_{X_1 X_2}(r_0) (M\nu_s)^{\frac{2}{n}} \\ &\quad + D(Q_c, X_i^n). \end{aligned} \quad (64)$$

In order to evaluate the quantity $D(Q_c, X_i^n)$ at high resolution, we can directly use Lemma 1 in [41], which leads to

$$D(Q_c, X_i^n) = G_c \nu_c^{\frac{2}{n}} (1 + o(1)) \text{ as } \nu_c \rightarrow 0. \quad (65)$$

Furthermore, in order to evaluate the rate, we need the following notation, for $i = 1, 2$,

$$\mathcal{P}_i \triangleq \mathbb{P}[Q_{in}(Q_c(X_i^n)) \in \mathcal{C}],$$

where \mathcal{C} is defined in (42). We will use the following lemma, which is proved in Appendix D.

Lemma 3. For $i = 1, 2$, we have $\lim_{(22)} \mathcal{P}_i = 0$.

Proof of Theorem 2: Relation (23) is proved in Appendix C. Relation (24) follows based on (64) and (65). Let us prove now equality (25). For this notice that the rate used to transmit $\beta_i(\lambda_i)$ is $\frac{1}{n}H(Q_{\mathcal{A}_i}(X_i^n))$. The rate needed for b_i is $\frac{1}{n}(-(1 - \mathcal{P}_i)\log_2(1 - \mathcal{P}_i) - \mathcal{P}_i\log_2\mathcal{P}_i)$. The rate used for encoding τ_i equals $\frac{1}{n}\mathcal{P}_i\log_2|\mathcal{T}| = \mathcal{P}_i$. Finally, the rate needed for encoding u_1 and u_2 using Slepian-Wolf coding equals $\frac{1}{n}H(U_1, U_2)$. As a consequence,

$$R(\mathcal{L}^{(n, r_0)}) = \frac{1}{n} \sum_{i=1}^2 [H(Q_{\mathcal{A}_i}(X_i^n)) - (1 - \mathcal{P}_i)\log_2(1 - \mathcal{P}_i) + \mathcal{P}_i(-\log_2\mathcal{P}_i + n)] + \frac{1}{n}H(U_1, U_2). \quad (66)$$

Since $\lim_{(22)} \bar{r}(\mathcal{A}_i(\mathbf{0})) = 0$, as shown in the proof of relation (23), we can apply Lemma 4 stated at the end of this appendix, which is due to Csiszar [42]. Thus, using the fact that $\nu(\mathcal{A}_i(\mathbf{0})) = \nu_s$, we obtain that

$$\lim_{(22)} \frac{1}{n} (H(Q_{\mathcal{A}_i}(X_i^n)) + \log_2(\nu_s)) = h(X_i). \quad (67)$$

Equations (66), (67) and Lemma 3 imply that

$$\lim_{(22)} \left(R(\mathcal{L}^{(n, r_0)}) + \frac{2}{n} \log_2(\nu_s) - \frac{1}{n}H(U_1, U_2) \right) = h(X_1) + h(X_2).$$

Relation (25) follows using the following equality, which is proved in Appendix C,

$$\lim_{(22)} H(U_i) = \log_2 K, \text{ for } i = 1, 2. \quad (68)$$

Further, inequality (26) is based on $H(U_2|U_1) \leq H(U_2) = \log_2 K$, while inequality (27) is proved in Appendix C. Finally, the claim that, in each of relations (23)-(25) and (27), the term hidden in the little-o notation can be upperbounded by a function which does not depend on the joint pdf $f_{X_1 X_2}$ and approaches 0 under (22) follows from the proofs of the aforementioned relations. ■

Proof of Corollary 1: Notice that $(M\nu_s)^{\frac{2}{n}} = (M^2 K \nu_c)^{\frac{2}{n}}$. By plugging (28) in (24) and using the fact that K and κ_0 are constants, relation (29) follows. Further, equalities (23) and (29) imply that

$$\begin{aligned} d_{s,i}(\mathcal{L}^{(n, r_0)}) d_{c,i}(\mathcal{L}^{(n, r_0)}) &= \frac{1}{4} G_{s/2} G_c (M\nu_s \nu_c)^{\frac{2}{n}} (1 + o(1)) \\ &= \frac{1}{4} G_{s/2} G_c \left(\frac{\nu_s^2}{K} \right)^{\frac{2}{n}} (1 + o(1)). \end{aligned}$$

By substituting this in (25), relation (30) follows.

In order to prove (31), we first apply Fano's inequality and obtain that

$$H(U_2|U_1) \leq H_b(\mathbb{P}[U_1 \neq U_2]) + \mathbb{P}[U_1 \neq U_2] \log_2 K, \quad (69)$$

where $H_b(\cdot)$ denotes the binary entropy function. Next we assume that $r_0 \leq r_c$ and use the following inequality proved in Appendix C (in the proof of relation (27))

$$\mathbb{P}[U_1 \neq U_2] \leq 1 - \left(1 - \frac{r_0}{r_c}\right)^n + \mathcal{P}_{X_1 X_2}(r_0) + o(1),$$

where the term hidden in the little-o notation does not depend on the joint pdf $f_{X_1 X_2}$. The fact that $\lim_{(22)} \frac{r_0}{r_c} = 0$, together with $\lim_{(22)} \mathcal{P}_{X_1 X_2}(r_0) = 0$, further implies that $\lim_{(22)} \mathbb{P}[U_1 \neq U_2] = 0$. Combining this with (69) leads to $\lim_{(22)} H(U_2|U_1) = 0$. By applying this result in (30), relation (31) follows. ■

Lemma 4 (Csiszar [42]). ³ Let $Z = (Z_1, \dots, Z_k)$ be an \mathbb{R}^k valued random vector with density f_Z . Suppose that there exists some Borel measurable partition $\mathcal{B}_0 = \{B_1, B_2, \dots\}$ of \mathbb{R}^k into sets of finite Lebesgue measure such that

$$-\sum_n \mathbb{P}[Z \in B_n] \log \mathbb{P}[Z \in B_n] < \infty.$$

Suppose furthermore, that for some $\rho > 0$, some positive integer s , and for all k , the distance of B_k from any other B_l is greater than ρ for all but at most s indexes l . Let $\mathcal{A} = \{A_0, A_1, \dots\}$ be a measurable partition with equal Lebesgue measure, i.e., $\lambda(A_i) = \epsilon$, $i = 1, 2, \dots$, and let us denote the supremum of the diameters of the sets A_i by $\delta(\mathcal{A})$. Then we have

$$\lim_{\delta(\mathcal{A}) \rightarrow 0} (H_{\mathcal{A}}(Z) + \log \epsilon) = h(f_Z),$$

where

$$H_{\mathcal{A}}(Z) = -\sum_n \mathbb{P}[Z \in A_n] \log \mathbb{P}[Z \in A_n],$$

and

$$h(f_Z) = -\int_{\mathbb{R}^k} f_Z(x^k) \log f_Z(x^k) dx^k,$$

the differential entropy of Z . Moreover, if Z has no density, then the above limit is $-\infty$. It should be mentioned that with the above conditions $h(f_Z)$ is always well-defined and $h(f_Z) < \infty$.

APPENDIX C

PROOF OF RELATIONS (23), (68) AND (27)

Proof of Relation (23): First let us fix i . We will split the proof into two parts. In Part 1 we show that if $\lim_{(22)} \frac{G(\mathcal{A}_i(\mathbf{0}))}{M^{\frac{2}{n}}}$ exists, then

$$\lim_{(22)} \frac{D(Q_{\mathcal{A}_i}, X_i^n)}{(M\nu_s)^{\frac{2}{n}}} = \lim_{(22)} \frac{G(\mathcal{A}_i(\mathbf{0}))}{M^{\frac{2}{n}}}. \quad (70)$$

In Part 2 we prove that

$$\lim_{(22)} \frac{G(\mathcal{A}_i(\mathbf{0}))}{M^{\frac{2}{n}}} = \frac{1}{4} G_{s/2}. \quad (71)$$

³The statement of this lemma is taken from [41].

Part 1.⁴ The proof is based on the idea that, in the limit of (22), the pdf $f_{X_i^n}$ can be approximated by a uniform density function over each set $\mathcal{A}_i(\lambda_s)$. This density function is $f_{\theta,\mu} : \mathbb{R}^n \rightarrow [0, \infty)$ defined as follows. For each $\lambda_s \in \Lambda_s$ and $x^n \in \mathcal{A}_i(\lambda_s)$, let

$$\begin{aligned} f_{\theta,\mu}(x^n) &= \frac{\mathbb{P}[X_i^n \in \mathcal{A}_i(\lambda_s)]}{\nu(\mathcal{A}_i(\lambda_s))} \\ &= \frac{1}{\nu(\mathcal{A}_i(\lambda_s))} \int_{\mathcal{A}_i(\lambda_s)} f_{X_i^n}(y^n) dy^n. \end{aligned}$$

Let $X_{\theta,\mu}^n$ denote the random variable with pdf $f_{\theta,\mu}$. Note that

$$\begin{aligned} &|D(Q_{\mathcal{A}_i}, X_{\theta,\mu}^n) - D(Q_{\mathcal{A}_i}, X_i^n)| \\ &\leq \frac{1}{n} \sum_{\lambda_s \in \Lambda_s} \int_{\mathcal{A}_i(\lambda_s)} \|x^n - \lambda_s\|^2 |f_{\theta,\mu}(x^n) - f_{X_i^n}(x^n)| dx^n \\ &\leq \frac{1}{n} \sum_{\lambda_s \in \Lambda_s} \bar{r}(\mathcal{A}_i(\mathbf{0}))^2 \int_{\mathcal{A}_i(\lambda_s)} |f_{\theta,\mu}(x^n) - f_{X_i^n}(x^n)| dx^n \\ &= \frac{\bar{r}(\mathcal{A}_i(\mathbf{0}))^2}{n} \int_{\mathbb{R}^n} |f_{\theta,\mu}(x^n) - f_{X_i^n}(x^n)| dx^n, \end{aligned} \quad (72)$$

where the second inequality is based on the fact that $\mathcal{A}_i(\lambda_s) = \lambda_s + \mathcal{A}_i(\mathbf{0})$, which implies that $\max_{x^n \in \mathcal{A}_i(\lambda_s)} \|x^n - \lambda_s\|^2 = \bar{r}(\mathcal{A}_i(\mathbf{0}))$. Let us analyze now the quantity $\bar{r}(\mathcal{A}_i(\mathbf{0}))$. Recall that $\mathcal{A}_i(\mathbf{0}) = \cup_{\lambda \in \beta_i^{-1}(\mathbf{0})} (\lambda + \mathcal{M}(\mathbf{0}))$, where $\mathcal{M}(\mathbf{0}) \triangleq \cup_{\lambda_c \in V_{in}(\mathbf{0}) \cap \Lambda_c} V_c(\lambda_c)$. Then it follows that

$$\bar{r}(\mathcal{A}_i(\mathbf{0})) \leq \bar{r}(\beta_i^{-1}(\mathbf{0})) + \bar{r}(\mathcal{M}(\mathbf{0})). \quad (73)$$

Further,

$$\bar{r}(\mathcal{M}(\mathbf{0})) \leq \bar{r}_{in} + \bar{r}_c \leq 2\bar{r}_{in} = 2\theta\bar{r}_{in,0}. \quad (74)$$

Since we are interested in computing the limits in (70) under (22), we may assume that μ is conveniently large. In particular, in the sequel we will assume that $\mu \geq 8$ so that relation (55) leads to

$$\bar{r}(\beta_i^{-1}(\mathbf{0})) \leq \mu^2 \theta \bar{r}_{in,0}. \quad (75)$$

Finally, relations (73)-(75), together with the fact that $M = \mu^n$ and $\nu_s = \mu^n \theta^n \nu_{in,0}$, lead to

$$\frac{\bar{r}(\mathcal{A}_i(\mathbf{0}))}{(M\nu_s)^{\frac{1}{n}}} \leq \frac{2\theta\bar{r}_{in,0} + \mu^2\theta\bar{r}_{in,0}}{\mu^2\theta\nu_{in,0}^{\frac{1}{n}}} \rightarrow \frac{\bar{r}_{in,0}}{2\nu_{in,0}^{\frac{1}{n}}} \quad (76)$$

in the limit of (22). The above result also implies that $\bar{r}(\mathcal{A}_i(\mathbf{0})) \rightarrow 0$ under (22). This enables us to apply Lemma 5, which is stated and proved in Appendix D, and we obtain that $f_{\theta,\mu}(x^n) \rightarrow f_{X_i^n}^n(x^n)$, $x^n \in \mathbb{R}^n$, under (22). Using further Scheffe's theorem [45], it follows that $\int_{\mathbb{R}^n} |f_{\theta,\mu}(x^n) - f_{X_i^n}(x^n)| dx^n \rightarrow 0$ under (22). Combining this further with (72) and (76) gives

$$\lim_{(22)} \frac{1}{(M\nu_s)^{\frac{2}{n}}} |D(Q_{\mathcal{A}_i}, X_{\theta,\mu}^n) - D(Q_{\mathcal{A}_i}, X_i^n)| = 0. \quad (77)$$

⁴This proof uses ideas from the proof of [41, Lemma 1].

Using now the fact that $f_{\theta,\mu}$ is uniform over each quantizer cell $\mathcal{A}_i(\lambda_s)$, we obtain that

$$\begin{aligned} D(Q_{\mathcal{A}_i}, X_{\theta,\mu}^n) &= \frac{1}{n} \sum_{\lambda_s \in \Lambda_s} \int_{\mathcal{A}_i(\lambda_s)} \|x^n - \lambda_s\|^2 f_{\theta,\mu}(x^n) dx^n \\ &= \frac{1}{n} \sum_{\lambda_s \in \Lambda_s} \frac{\mathbb{P}[X_i^n \in \mathcal{A}_i(\lambda_s)]}{\nu(\mathcal{A}_i(\lambda_s))} \int_{\mathcal{A}_i(\lambda_s)} \|x^n - \lambda_s\|^2 dx^n \\ &\stackrel{(a)}{=} \frac{1}{n\nu(\mathcal{A}_i(\mathbf{0}))} \int_{\mathcal{A}_i(\mathbf{0})} \|x^n\|^2 dx^n \sum_{\lambda_s \in \Lambda_s} \mathbb{P}[X_i^n \in \mathcal{A}_i(\lambda_s)] \\ &= \frac{1}{n\nu(\mathcal{A}_i(\mathbf{0}))} \int_{\mathcal{A}_i(\mathbf{0})} \|x^n\|^2 dx^n \mathbb{P}[X_i^n \in \mathbb{R}^n] \\ &= G(\mathcal{A}_i(\mathbf{0})) (\nu(\mathcal{A}_i(\mathbf{0})))^{\frac{2}{n}} \\ &\stackrel{(b)}{=} G(\mathcal{A}_i(\mathbf{0})) \nu_s^{\frac{2}{n}}, \end{aligned} \quad (78)$$

where (a) uses the fact that $\mathcal{A}_i(\lambda_s) = \lambda_s + \mathcal{A}_i(\mathbf{0})$, while (b) is based on the fact that $\nu(\mathcal{A}_i(\mathbf{0})) = \nu_s$ since $\mathcal{A}_i(\mathbf{0})$ is a fundamental cell of the lattice Λ_s . Relations (77) and (78) prove the claim of Part 1.

Part 2. In order to prove (71), we will first evaluate $\int_{\mathcal{A}_i(\mathbf{0})} \|x^n\|^2 dx^n$. Using the fact that $\mathcal{A}_i(\mathbf{0}) = \cup_{\lambda \in \beta_i^{-1}(\mathbf{0})} (\lambda + \mathcal{M}(\mathbf{0}))$ and relation (41), we obtain that

$$\mathcal{A}_i(\mathbf{0}) = \cup_{\lambda \in \mathcal{U}} (\lambda - \beta_i(\lambda) + \mathcal{M}(\mathbf{0})). \quad (79)$$

Using further Lemma 6, which is stated and proved in Appendix D, we obtain that

$$\begin{aligned} \int_{\lambda - \beta_i(\lambda) + \mathcal{M}(\mathbf{0})} \|x^n\|^2 dx^n &= \|\lambda - \beta_i(\lambda)\|^2 \nu(\mathcal{M}(\mathbf{0})) \\ &+ 2 \langle \int_{\mathcal{M}(\mathbf{0})} x^n dx^n, \lambda - \beta_i(\lambda) \rangle + \int_{\mathcal{M}(\mathbf{0})} \|x^n\|^2 dx^n. \end{aligned} \quad (80)$$

It is easy to see that $\mathcal{M}(\mathbf{0})$ is a fundamental cell of the lattice Λ_{in} , therefore, $\nu(\mathcal{M}(\mathbf{0})) = \nu_{in}$. Further, relations (79) and (80) lead to

$$\begin{aligned} \int_{\mathcal{A}_i(\mathbf{0})} \|x^n\|^2 dx^n &= |\mathcal{U}| \underbrace{\int_{\mathcal{M}(\mathbf{0})} \|x^n\|^2 dx^n}_{T_1} \\ &+ 2 \underbrace{\sum_{\lambda \in \mathcal{U}} \langle \int_{\mathcal{M}(\mathbf{0})} x^n dx^n, \lambda - \beta_i(\lambda) \rangle}_{T_{2,i}} \\ &+ \nu_{in} \underbrace{\sum_{\lambda \in \mathcal{U}} \|\lambda - \beta_i(\lambda)\|^2}_{T_{3,i}}. \end{aligned}$$

Then

$$\begin{aligned} \frac{G(\mathcal{A}_i(\mathbf{0}))}{M^{\frac{2}{n}}} &= \frac{T_1}{nM^{\frac{2}{n}}(M\nu_{in})^{1+\frac{2}{n}}} + \frac{T_{2,i}}{nM^{\frac{2}{n}}(M\nu_{in})^{1+\frac{2}{n}}} \\ &+ \frac{T_{3,i}}{nM^{\frac{2}{n}}(M\nu_{in})^{1+\frac{2}{n}}}. \end{aligned} \quad (81)$$

We will prove first that the first two terms on the right hand side of the above equality approach 0 in the limit of (22). Consider the first term. Note that $\int_{\mathcal{M}(\mathbf{0})} \|x^n\|^2 dx^n \leq$

$(\bar{r}(\mathcal{M}(\mathbf{0})))^2 \nu_{in}$. Combining this further with (74) and the fact that $|\mathcal{U}| = M$ gives

$$\begin{aligned} \frac{T_1}{nM^{\frac{2}{n}}(M\nu_{in})^{1+\frac{2}{n}}} &\leq \frac{4M\theta^2\bar{r}_{in,0}^2\nu_{in}}{nM^{\frac{2}{n}}(M\nu_{in})^{1+\frac{2}{n}}} \\ &= \frac{4\bar{r}_{in,0}^2}{nM^{\frac{4}{n}}\nu_{in,0}^{\frac{2}{n}}} \rightarrow 0 \text{ under (22)}. \end{aligned} \quad (82)$$

It is easy to see that the closure of a lattice Voronoi cell of the origin is symmetric about the origin. Therefore, if Λ_{in} is a clean sublattice of Λ_c , i.e., there are no points of Λ_c on the boundary of $V_{in}(\mathbf{0})$, then the set $\Lambda_c \cap V_{in}(\mathbf{0})$ is symmetric about the origin. The above considerations further imply that the closure of the set $\mathcal{M}(\mathbf{0})$ is symmetric about the origin, thus $\int_{\mathcal{M}(\mathbf{0})} x^n dx^n = 0$. Then the second term in (81) is 0. When Λ_{in} is not a clean sublattice of Λ_c , the aforementioned term still approaches 0 in the limit of (22), as we prove next. Note that

$$\begin{aligned} |T_{2,i}| &= 2 \left| \sum_{\lambda \in \mathcal{U}} \int_{\mathcal{M}(\mathbf{0})} \langle x^n, \lambda - \beta_i(\lambda) \rangle dx^n \right| \\ &\leq 2 \sum_{\lambda \in \mathcal{U}} \int_{\mathcal{M}(\mathbf{0})} |\langle x^n, \lambda - \beta_i(\lambda) \rangle| dx^n \\ &\stackrel{(a)}{\leq} 2 \sum_{\lambda \in \mathcal{U}} \int_{\mathcal{M}(\mathbf{0})} \|x^n\| \|\lambda - \beta_i(\lambda)\| dx^n \\ &= 2 \int_{\mathcal{M}(\mathbf{0})} \|x^n\| dx^n \sum_{\lambda \in \mathcal{U}} \|\lambda - \beta_i(\lambda)\| \\ &\stackrel{(b)}{\leq} 2\bar{r}(\mathcal{M}(\mathbf{0}))\nu_{in}M(\max_{\lambda \in \mathcal{U}} \|\lambda\| + \max_{\lambda \in \mathcal{U}} \|\beta_i(\lambda)\|) \\ &\stackrel{(c)}{\leq} 4\theta\bar{r}_{in,0}\nu_{in}M\mu^2\theta\bar{r}_{in,0} \stackrel{(d)}{=} 4\theta^2\nu_{in}M^{1+\frac{2}{n}}\bar{r}_{in,0}^2. \end{aligned} \quad (83)$$

Here (a) follows from the Cauchy-Schwarz inequality and (b) is based on the fact that $\int_{\mathcal{M}(\mathbf{0})} \|x^n\| dx^n \leq \bar{r}(\mathcal{M}(\mathbf{0}))\nu_{in}$ and $|\mathcal{U}| = M$; additionally, (c) follows from (74) and the discussion in the paragraph below equation (56); finally, (d) is based on the fact that $\mu = M^{\frac{1}{n}}$. Further, relation (83) implies that

$$\begin{aligned} \frac{|T_{2,i}|}{nM^{\frac{2}{n}}(M\nu_{in})^{1+\frac{2}{n}}} &\leq \frac{4\theta^2\nu_{in}M^{1+\frac{2}{n}}\bar{r}_{in,0}^2}{nM^{1+\frac{4}{n}}\nu_{in}\theta^2\nu_{in,0}^{\frac{2}{n}}} \\ &= \frac{4\bar{r}_{in,0}^2}{nM^{\frac{2}{n}}\nu_{in,0}^{\frac{2}{n}}} \rightarrow 0 \text{ under (22)}. \end{aligned} \quad (84)$$

Let us evaluate now $\frac{T_{3,i}}{\nu_{in}}$. We need to treat separately the cases $i = 1$ and $i = 2$. Recall that $\mathcal{U} = \cup_{\tau \in \mathcal{T}} V_{s/2}(\tau) \cap \Lambda_{in}$. We will denote $\hat{V}_{s/2}(\tau) \triangleq V_{s/2}(\tau) \cap \Lambda_{in}$. Using further (39), we obtain that

$$\begin{aligned} \frac{T_{3,1}}{\nu_{in}} &= \sum_{\tau \in \mathcal{T}} \sum_{\lambda \in \hat{V}_{s/2}(\tau)} \|\lambda - \mu(\lambda - \tau)\|^2 \\ &= \sum_{\tau \in \mathcal{T}} \sum_{\lambda \in \hat{V}_{s/2}(\tau)} \|(1 - \mu)(\lambda - \tau) + \tau\|^2 \\ &= \sum_{\tau \in \mathcal{T}} \sum_{\lambda \in \hat{V}_{s/2}(\tau)} (\|(1 - \mu)(\lambda - \tau)\|^2 + \|\tau\|^2) \end{aligned}$$

$$\begin{aligned} &+ 2\langle (1 - \mu)(\lambda - \tau), \tau \rangle \\ &= \sum_{\tau \in \mathcal{T}} \sum_{\lambda \in \hat{V}_{s/2}(\tau)} (1 - \mu)^2 \|\lambda - \tau\|^2 + \sum_{\tau \in \mathcal{T}} \sum_{\lambda \in \hat{V}_{s/2}(\tau)} \|\tau\|^2 \\ &+ 2 \underbrace{\sum_{\tau \in \mathcal{T}} \sum_{\lambda \in \hat{V}_{s/2}(\tau)} \langle (1 - \mu)(\lambda - \tau), \tau \rangle}_{T_4} \\ &\stackrel{(a)}{=} \underbrace{(1 - \mu)^2 |\mathcal{T}| \sum_{\lambda \in \hat{V}_{s/2}(\mathbf{0})} \|\lambda\|^2}_{T_5} + \underbrace{\frac{M}{|\mathcal{T}|} \sum_{\tau \in \mathcal{T}} \|\tau\|^2}_{T_6} + T_4, \end{aligned} \quad (85)$$

where (a) is based on the fact that $\hat{V}_{s/2}(\tau) = \tau + \hat{V}_{s/2}(\mathbf{0})$ and $|\hat{V}_{s/2}(\mathbf{0})| = \frac{M}{|\mathcal{T}|}$. Relation (85) leads to

$$\begin{aligned} \frac{T_{3,1}}{nM^{\frac{2}{n}}(M\nu_{in})^{1+\frac{2}{n}}} &= \frac{T_4}{nM^{1+\frac{4}{n}}\nu_{in}^{\frac{2}{n}}} + \frac{T_5}{nM^{1+\frac{4}{n}}\nu_{in}^{\frac{2}{n}}} + \frac{T_6}{nM^{1+\frac{4}{n}}\nu_{in}^{\frac{2}{n}}}. \end{aligned} \quad (86)$$

We will show first that the first and last terms on the right hand side of (86) approach 0 in the limit of (22). For this we need to introduce the following notation. For any two nested lattices $\Lambda_2 \subset \Lambda_1$ in \mathbb{R}^n , denote $\mathcal{C}_{\Lambda_2:\Lambda_1} \triangleq \cup_{\lambda_1 \in V_{\Lambda_2}(\mathbf{0}) \cap \Lambda_1} V_{\Lambda_1}(\lambda_1)$. Using Lemma 7, which is stated and proved in Appendix D, we obtain

$$\begin{aligned} \frac{T_6}{nM^{1+\frac{4}{n}}\nu_{in}^{\frac{2}{n}}} &= \frac{\frac{M}{2^n}n2^n}{nM^{1+\frac{4}{n}}\nu_{in}^{\frac{2}{n}}} \left(G(\mathcal{C}_{\Lambda_s:\Lambda_{s/2}})\nu_s^{\frac{2}{n}} - G_{s/2}\nu_{s/2}^{\frac{2}{n}} \right) \\ &= \frac{1}{M^{\frac{2}{n}}} \left(G(\mathcal{C}_{\Lambda_s:\Lambda_{s/2}}) - \frac{1}{4}G_{s/2} \right), \end{aligned}$$

where the last equality is based on $\nu_s = M\nu_{in}$ and $\nu_{s/2} = M\nu_{in}/2^n$. As the parameters μ and θ vary, both lattices Λ_s and $\Lambda_{s/2}$ are scaled by the same factor, therefore the set $\mathcal{C}_{\Lambda_s:\Lambda_{s/2}}$ is scaled by that factor. Since the second moment is invariant under scaling, it follows that $G(\mathcal{C}_{\Lambda_s:\Lambda_{s/2}}) - \frac{1}{4}G_{s/2}$ remains constant as θ and μ vary. Consequently,

$$\lim_{(22)} \frac{T_6}{nM^{1+\frac{4}{n}}\nu_{in}^{\frac{2}{n}}} = 0. \quad (87)$$

Consider now the first term on the right hand side of (86). We have

$$\begin{aligned} |T_4| &\leq 2|\mu - 1| \sum_{\tau \in \mathcal{T}} \sum_{\lambda \in \hat{V}_{s/2}(\tau)} |\langle \lambda - \tau, \tau \rangle| \\ &\stackrel{(a)}{\leq} 2|\mu - 1| \sum_{\tau \in \mathcal{T}} \sum_{\lambda \in \hat{V}_{s/2}(\tau)} \|\lambda - \tau\| \|\tau\| \\ &\leq 2|\mu - 1| M \max_{\tau \in \mathcal{T}} \|\tau\| \max_{\lambda \in \hat{V}_{s/2}(\mathbf{0})} \|\lambda\| \\ &\leq 2\mu M \bar{r}_s \bar{r}_{s/2}, \end{aligned}$$

where (a) is based on the Cauchy-Schwarz inequality. Using further the fact that $\mu = M^{\frac{1}{n}}$, while $\bar{r}_{s/2} = \bar{r}_s/2 = M^{\frac{1}{n}}\bar{r}_{in}/2$,

leads to

$$\begin{aligned} \frac{|T_4|}{nM^{1+\frac{4}{n}}\nu_{in}^{\frac{2}{n}}} &\leq \frac{M^{1+\frac{3}{n}}\bar{r}_{in}^2}{nM^{1+\frac{4}{n}}\nu_{in}^{\frac{2}{n}}} \\ &= \frac{\bar{r}_{in,0}^2}{nM^{\frac{1}{n}}\nu_{in,0}^{\frac{2}{n}}} \rightarrow 0 \text{ as (22) holds.} \end{aligned} \quad (88)$$

In order to evaluate the second term in (86), we use again Lemma 7 and obtain that

$$\begin{aligned} \frac{T_5}{nM^{1+\frac{4}{n}}\nu_{in}^{\frac{2}{n}}} &= \frac{(\mu-1)^2 nM}{nM^{1+\frac{4}{n}}\nu_{in}^{\frac{2}{n}}} \left(G(\mathcal{C}_{\Lambda_{s/2}:\Lambda_{in}})\nu_{s/2}^{\frac{2}{n}} - G_{in}\nu_{in}^{\frac{2}{n}} \right) \\ &= \frac{(M^{\frac{1}{n}}-1)^2}{M^{\frac{4}{n}}} \left(G(\mathcal{C}_{\Lambda_{s/2}:\Lambda_{in}})\frac{M^{\frac{2}{n}}}{4} - G_{in} \right), \end{aligned}$$

where the last equality relies on the fact that $\mu = M^{\frac{1}{n}}$, while $\nu_{s/2} = M\nu_{in}/2^n$. Further, we obtain that

$$\lim_{(22)} \frac{T_5}{nM^{1+\frac{4}{n}}\nu_{in}^{\frac{2}{n}}} = \lim_{(22)} \frac{G(\mathcal{C}_{\Lambda_{s/2}:\Lambda_{in}})}{4} = \frac{G_{s/2}}{4}, \quad (89)$$

where the last equality follows from Lemma 8, which is stated and proved in Appendix D.

Relations (86)-(89) imply that

$$\lim_{(22)} \frac{T_{3,1}}{nM^{1+\frac{4}{n}}\nu_{in}^{\frac{2}{n}+1}} = \frac{1}{4}G_{s/2}. \quad (90)$$

Combining the above with (81), (82) and (84), we obtain that (71) holds for $i = 1$. In order to prove the claim for $i = 2$, we need to evaluate now $\frac{T_{3,2}}{\nu_{in}}$. Note that

$$\begin{aligned} \frac{T_{3,2}}{\nu_{in}} &= \sum_{\tau \in \mathcal{T}} \sum_{\lambda \in \hat{V}_{s/2}(\tau)} \|\lambda - 2\tau + \mu(\lambda - \tau)\|^2 \\ &= \sum_{\tau \in \mathcal{T}} \sum_{\lambda \in \hat{V}_{s/2}(\tau)} \|(1+\mu)(\lambda - \tau) - \tau\|^2 \\ &= \sum_{\tau \in \mathcal{T}} \sum_{\lambda \in \hat{V}_{s/2}(\tau)} (\|(1+\mu)(\lambda - \tau)\|^2 + \|\tau\|^2 \\ &\quad + 2\langle (1+\mu)(\lambda - \tau), \tau \rangle) \\ &= \sum_{\tau \in \mathcal{T}} \sum_{\lambda \in \hat{V}_{s/2}(\tau)} (\|(1+\mu)(\lambda - \tau)\|^2 + \|\tau\|^2) \\ &\quad + 2 \sum_{\tau \in \mathcal{T}} \sum_{\lambda \in \hat{V}_{s/2}(\tau)} \langle (1+\mu)(\lambda - \tau), \tau \rangle. \end{aligned}$$

Next the conclusion follows using similar arguments as for $i = 1$. This observation concludes the proof. \blacksquare

Proof of Relation (68): In order to prove the claim, we will show that U_i approaches a uniform distribution. To prove this let $u \in V_{in}(\mathbf{0}) \cap \Lambda_c$.

The general idea of the proof is that, as the limits of (22) are approached, the pdf $f_{X_i^n}$ can be approximated by a pdf

which is uniform on each set $\mathcal{M}(\lambda)$. Then in the limit of (22),

$$\begin{aligned} \mathbb{P}[U_i = u] &= \sum_{\lambda \in \Lambda_{in}} \int_{V_c(\lambda+u)} f_{X_i^n}(x^n) dx^n \\ &\stackrel{(a)}{\approx} \sum_{\lambda \in \Lambda_{in}} f_{X_i^n}(\lambda) \nu_c \\ &= \sum_{\lambda \in \Lambda_{in}} f_{X_i^n}(\lambda) \frac{\nu_{in}}{K} \\ &\stackrel{(b)}{\approx} \frac{1}{K} \sum_{\lambda \in \Lambda_{in}} \int_{\mathcal{M}(\lambda)} f_{X_i^n}(x^n) dx^n \\ &= \frac{1}{K}. \end{aligned}$$

Next we provide a rigorous treatment of relations (a) and (b).

Define a density function $f_{\theta,\mu} : \mathbb{R}^n \rightarrow [0, \infty)$, which is uniform on each set $\mathcal{M}(\lambda)$, as follows

$$f_{\theta,\mu}(x^n) = \frac{1}{\nu(\mathcal{M}(\lambda))} \int_{\mathcal{M}(\lambda)} f_{X_i^n}(y^n) dy^n, \quad x^n \in \mathcal{M}(\lambda).$$

Then in view of Lemma 5 (stated and proved in Appendix D), we have that $f_{\theta,\mu}(x^n) \rightarrow f_{X_i^n}(x^n)$, $x^n \in \mathbb{R}^n$, under (22).

Further, we have

$$\begin{aligned} \mathbb{P}[U_i = u] &= \int_{\cup_{\lambda \in \Lambda_{in}} V_c(\lambda+u)} (f_{X_i^n}(x^n) - f_{\theta,\mu}(x^n)) dx^n \\ &\quad + \int_{\cup_{\lambda \in \Lambda_{in}} V_c(\lambda+u)} f_{\theta,\mu}(x^n) dx^n \\ &\leq \int_{\cup_{\lambda \in \Lambda_{in}} V_c(\lambda+u)} |f_{X_i^n}(x^n) - f_{\theta,\mu}(x^n)| dx^n \\ &\quad + \sum_{\lambda \in \Lambda_{in}} \int_{V_c(\lambda+u)} f_{\theta,\mu}(x^n) dx^n. \end{aligned} \quad (91)$$

Note that

$$\begin{aligned} &\int_{\cup_{\lambda \in \Lambda_{in}} V_c(\lambda+u)} |f_{X_i^n}(x^n) - f_{\theta,\mu}(x^n)| dx^n \\ &\leq \int_{\mathbb{R}^n} |f_{X_i^n}(x^n) - f_{\theta,\mu}(x^n)| dx^n \rightarrow 0 \text{ under (22),} \end{aligned}$$

where the last relation is valid in view of Scheffe's theorem [45].

Further, since $f_{\theta,\mu}$ is constant on each $\mathcal{M}(\lambda)$, we have

$$\begin{aligned} \sum_{\lambda \in \Lambda_{in}} \int_{V_c(\lambda+u)} f_{\theta,\mu}(x^n) dx^n &= \sum_{\lambda \in \Lambda_{in}} f_{\theta,\mu}(\lambda) \frac{\nu_{in}}{K} \\ &= \frac{1}{K} \sum_{\lambda \in \Lambda_{in}} \int_{\mathcal{M}(\lambda)} f_{X_i^n}(x^n) dx^n \\ &= \frac{1}{K}. \end{aligned} \quad (92)$$

Relations (91)-(92), together with the fact that the size of the alphabet of U_i is K and K is constant, prove the claim. With this observation the proof is complete. \blacksquare

Proof of Relation (27): Using a variant of Fano's inequality, we obtain that

$$H(U_2|U_1) \leq 1 + \mathbb{P}[U_1 \neq U_2] \log_2 K, \quad (93)$$

where we used the fact that $H(U_2) = \log_2 K$. Let $\lambda_{c,1} = Q_c(x_1^n)$. Notice that if $x_2^n - x_1^n \in \mathcal{B}(r_0)$ and the distance from

x_1^n to the boundary of the Voronoi cell $V_c(\lambda_{c,1})$ is larger than or equal to r_0 , then it is guaranteed that $x_2^n \in V_c(\lambda_{c,1})$, thus $u_2 = u_1$. Now let us denote

$$\mathcal{E}(\lambda_c) \triangleq V_c(\lambda_c) \setminus \left(1 - \frac{r_0}{r_c}\right) V_c(\lambda_c),$$

for each $\lambda_c \in \Lambda_c$, and $\mathcal{E} \triangleq \cup_{\lambda_c \in \Lambda_c} \mathcal{E}(\lambda_c)$. It follows that

$$\mathbb{P}[U_1 \neq U_2] \leq \mathcal{P}_{X_1 X_2}(r_0) + \mathbb{P}[X_1^n \in \mathcal{E}]. \quad (94)$$

Further, we obtain

$$\mathbb{P}[X_1^n \in \mathcal{E}] \leq \int_{\mathcal{E}} |f_{X_1^n}(x^n) - f_{\theta,c}(x^n)| dx^n + \int_{\mathcal{E}} f_{\theta,c}(x^n) dx^n, \quad (95)$$

where $f_{\theta,\mu}$ was defined in the proof of relation (68). According to that proof, the first integral in (95) approaches 0 in the limit of (22). Since $f_{\theta,\mu}$ is uniform over each Voronoi region of the central lattice, we have

$$\begin{aligned} \int_{\mathcal{E}} f_{\theta,\mu}(x^n) dx^n &= \sum_{\lambda_c \in \Lambda_c} \int_{\mathcal{E}(\lambda_c)} f_{\theta,\mu}(x^n) dx^n \\ &= \sum_{\lambda_c \in \Lambda_c} f_{\theta,\mu}(\lambda_c) \nu(\mathcal{E}(\lambda_c)) \\ &= \left(1 - \left(1 - \frac{r_0}{r_c}\right)^n\right) \sum_{\lambda_c \in \Lambda_c} f_{\theta,\mu}(\lambda_c) \nu_c \\ &= 1 - \left(1 - \frac{r_0}{r_c}\right)^n. \end{aligned} \quad (96)$$

Relations (95)-(96), together with the fact that the first integral in (95) approaches 0 in the limit of (22), imply that

$$\mathbb{P}[U_1 \neq U_2] \leq 1 - \left(1 - \frac{r_0}{r_c}\right)^n + \mathcal{P}_{X_1 X_2}(r_0) + o(1).$$

Finally, by applying the above inequality in (93), the conclusion follows. ■

APPENDIX D PROOFS OF LEMMAS

Proof of Lemma 2: Throughout the proof we will use the fact that μ is an even integer and, consequently, $\mu \geq 2$, which implies that $\bar{r}_c, \bar{r}_{in} \leq 1/2\bar{r}_s$. Using further the fact that $\lambda_{c,i} = \lambda_i + u_i$ and the triangle inequality, we obtain that

$$\begin{aligned} \|x_i^n - \hat{x}_{c,i}^n\| &= \|x_i^n - \lambda_{c,i} + u_i + \lambda_i - \hat{x}_{c,i}^n\| \\ &\leq \|x_{c,i}^n - \lambda_{c,i}\| + \|u_i\| + \|\lambda_i - \hat{x}_{c,i}^n\| \\ &\leq \bar{r}_c + \bar{r}_{in} + \|\lambda_i - \hat{x}_{c,i}^n\| \\ &\leq \bar{r}_s + \|\lambda_i - \hat{x}_{c,i}^n\|. \end{aligned} \quad (97)$$

If condition (43) is violated, then $\hat{x}_{c,i}^n = \lambda_{s,i}$. Thus, we have

$$\|\lambda_i - \hat{x}_{c,i}^n\| = \|\lambda_i - \lambda_{s,i}\| \leq \bar{r}(\beta_i^{-1}(\mathbf{0})) \leq (4 + \mu/2)\bar{r}_s,$$

where the last inequality is from (55). The above relations, together with (97), imply that

$$\|x_i^n - \hat{x}_{c,i}^n\| \leq (5 + \mu/2)\bar{r}_s \leq 3c\bar{r}_s,$$

proving that the claim holds when (43) is not true.

Let us assume now that condition (43) is satisfied and that Case 3) holds at the decoder, i.e., $b_1 = b_2 = 1$ and $\tau_1 \neq \tau_2$. Thus, $\hat{x}_{c,i}^n = \tilde{\lambda}_i + u_i$, where $\tilde{\lambda}_i$ is given in (53) and (54). Then

$$\|\lambda_i - \hat{x}_{c,i}^n\| \leq \|\lambda_i - \tilde{\lambda}_i\| + \|u_i\| \leq \|\lambda_i - \tilde{\lambda}_i\| + \bar{r}_{in}. \quad (98)$$

Let us consider now $i = 1$. Using (53), (54) and the triangle inequality, we obtain that

$$\begin{aligned} &\|\lambda_1 - \tilde{\lambda}_1\| \\ &\leq \|\lambda_1 - \tilde{\lambda}_s\| + \|\tau_1\| + \frac{1}{\mu} \|\lambda_{s,1} - \tilde{\lambda}_s\| \\ &\leq \|\lambda_1 - \lambda_{s,1}\| + \|\lambda_{s,1} - \tilde{\lambda}_s\| + \bar{r}_s + \frac{1}{\mu} \|\lambda_{s,1} - \tilde{\lambda}_s\| \\ &\leq (4 + \mu/2)\bar{r}_s + \bar{r}_s + \left(1 + \frac{1}{\mu}\right) \|\lambda_{s,1} - \tilde{\lambda}_s\|, \end{aligned} \quad (99)$$

where the last inequality is based on $\|\lambda_1 - \lambda_{s,1}\| \leq \bar{r}(\beta_i^{-1}(\mathbf{0}))$ and (55). Using now (52) in conjunction with the triangle inequality leads to

$$\begin{aligned} \|\lambda_{s,1} - \tilde{\lambda}_s\| &\leq \|\lambda_{s,1} - \tilde{v}\| + (\mu + 1)\|\tilde{w}\| \\ &\leq \|\lambda_{s,1} - \tilde{v}\| + 2(\mu + 1)\bar{r}_s, \end{aligned} \quad (100)$$

where the last inequality follows based on (51) and the fact that

$$\begin{aligned} \|\tilde{w}\| &\leq \|\hat{w} + \frac{1}{2}(\tau_2 - \tau_1) - Q_s(\hat{w} + \frac{1}{2}(\tau_2 - \tau_1))\| \\ &\quad + \left\|\frac{1}{2}(\tau_2 - \tau_1)\right\| \\ &\leq 2\bar{r}_s. \end{aligned}$$

Finally, based on (49) and (50), we obtain that

$$\begin{aligned} \|\lambda_{s,1} - \tilde{v}\| &= \left\|\frac{1}{2}(\lambda_{s,1} - \lambda_{s,2}) - \frac{1}{2}\mu\tilde{\lambda} + \left(1 + \frac{\mu}{2}\right)\tau_2 - \frac{\mu}{2}\tau_1\right\| \\ &\leq \frac{1}{2}\|(\lambda_{s,1} - \lambda_{s,2})\| + \frac{1}{2}\mu\|\tilde{\lambda}\| + \left(1 + \frac{\mu}{2}\right)\|\tau_2\| \\ &\quad + \frac{\mu}{2}\|\tau_1\|. \end{aligned} \quad (101)$$

Notice that relation (43) implies that

$$\|(\lambda_{s,1} - \lambda_{s,2})\| \leq (10 + \mu)\bar{r}_s. \quad (102)$$

Additionally, from (44) we obtain that

$$\begin{aligned} \|\tilde{\lambda}\| &\leq \|u_1 - u_2\| + \|(u_1 - u_2) - Q_{in}(u_1 - u_2)\| \\ &\leq 2\bar{r}_{in} + \bar{r}_{in} \leq 2\bar{r}_s. \end{aligned} \quad (103)$$

Plugging (102) and (103) in (101) leads to

$$\|\lambda_{s,1} - \tilde{v}\| \leq (5 + \mu/2)\bar{r}_s + \bar{r}_s + (1 + \mu)\bar{r}_s = (7 + 3\mu/2)\bar{r}_s.$$

The above relations and (100) imply that

$$\|\lambda_{s,1} - \tilde{\lambda}_s\| \leq (9 + 7\mu/2)\bar{r}_s.$$

Combining now the above inequality with (97), (98) and (99), we obtain that

$$\begin{aligned} \|x_1^n - \hat{x}_{c,1}^n\| &\leq \bar{r}_s + 1/2\bar{r}_s + (5 + \mu/2)\bar{r}_s \\ &\quad + \left(1 + \frac{1}{\mu}\right) (9 + 7\mu/2)\bar{r}_s \\ &\leq (24 + 4\mu)\bar{r}_s \\ &\leq 16\mu\bar{r}_s, \end{aligned}$$

Lemma 6. For any set $\mathcal{A} \subseteq \mathbb{R}^n$ and any $u \in \mathbb{R}^n$,

$$\begin{aligned} & \int_{u+\mathcal{A}} \|x^n\|^2 dx^n \\ &= \int_{\mathcal{A}} \|x^n\|^2 dx^n + 2 \langle \int_{\mathcal{A}} x^n dx^n, u \rangle + \|u\|^2 \nu(\mathcal{A}). \end{aligned}$$

Proof: Applying the change of variable $x^n = u + y^n$, we obtain that

$$\begin{aligned} & \int_{u+\mathcal{A}} \|x^n\|^2 dx^n \\ &= \int_{\mathcal{A}} \|y^n + u\|^2 dy^n \\ &= \int_{\mathcal{A}} \|y^n\|^2 dy^n + \int_{\mathcal{A}} 2 \langle y^n, u \rangle dy^n + \int_{\mathcal{A}} \|u\|^2 dy^n \\ &= \int_{\mathcal{A}} \|y^n\|^2 dy^n + 2 \langle \int_{\mathcal{A}} x^n dx^n, u \rangle + \|u\|^2 \nu(\mathcal{A}). \end{aligned}$$

Lemma 7. Let $\Lambda_2 \subset \Lambda_1$ be two nested lattices in \mathbb{R}^n . Let $N_0 \triangleq N(\Lambda_2 : \Lambda_1)$ and $\mathcal{C}_{\Lambda_2:\Lambda_1} \triangleq \cup_{\lambda_1 \in V_{\Lambda_2}(\mathbf{0}) \cap \Lambda_1} V_{\Lambda_1}(\lambda_1)$. Then

$$\sum_{\lambda_1 \in V_{\Lambda_2}(\mathbf{0}) \cap \Lambda_1} \|\lambda_1\|^2 = nN_0 \left(G(\mathcal{C}_{\Lambda_2:\Lambda_1}) \nu_{\Lambda_2}^{\frac{2}{n}} - G_{\Lambda_1} \nu_{\Lambda_1}^{\frac{2}{n}} \right).$$

Proof: It can be easily seen that $\mathcal{C}_{\Lambda_2:\Lambda_1}$ is a fundamental region of the lattice Λ_2 , thus $\nu(\mathcal{C}_{\Lambda_2:\Lambda_1}) = \nu(\Lambda_2) = N_0 \nu(\Lambda_1)$. Invoking further the definition of $G(\mathcal{C}_{\Lambda_2:\Lambda_1})$ gives

$$nN_0 G(\mathcal{C}_{\Lambda_2:\Lambda_1}) \nu_{\Lambda_2}^{\frac{2}{n}} = \frac{1}{\nu_{\Lambda_1}} \int_{\mathcal{C}_{\Lambda_2:\Lambda_1}} \|x^n\|^2 dx^n.$$

Using the fact that $V_{\Lambda_1}(\lambda_1) = \lambda_1 + V_{\Lambda_1}(\mathbf{0})$, we obtain that

$$\begin{aligned} & \frac{1}{\nu_{\Lambda_1}} \int_{\mathcal{C}_{\Lambda_2:\Lambda_1}} \|x^n\|^2 dx^n \\ &= \frac{1}{\nu_{\Lambda_1}} \sum_{\lambda_1 \in V_{\Lambda_2}(\mathbf{0}) \cap \Lambda_1} \int_{\lambda_1 + V_{\Lambda_1}(\mathbf{0})} \|x^n\|^2 dx^n \\ &\stackrel{(a)}{=} \frac{1}{\nu_{\Lambda_1}} \sum_{\lambda_1 \in V_{\Lambda_2}(\mathbf{0}) \cap \Lambda_1} \left(\int_{V_{\Lambda_1}(\mathbf{0})} \|x^n\|^2 dx^n \right. \\ &\quad \left. + 2 \left\langle \int_{V_{\Lambda_1}(\mathbf{0})} x^n dx^n, \lambda_1 \right\rangle + \|\lambda_1\|^2 \nu_{\Lambda_1} \right) \\ &\stackrel{(b)}{=} \frac{N_0}{\nu_{\Lambda_1}} \int_{V_{\Lambda_1}(\mathbf{0})} \|x^n\|^2 dx^n + \sum_{\lambda_1 \in V_{\Lambda_2}(\mathbf{0}) \cap \Lambda_1} \|\lambda_1\|^2 \\ &\stackrel{(c)}{=} nN_0 G_{\Lambda_1} \nu_{\Lambda_1}^{\frac{2}{n}} + \sum_{\lambda_1 \in V_{\Lambda_2}(\mathbf{0}) \cap \Lambda_1} \|\lambda_1\|^2. \end{aligned}$$

Here (a) is based on Lemma 6; moreover, (b) uses the fact that $\int_{V_{\Lambda_1}(\mathbf{0})} x^n dx^n = \mathbf{0}$ and $|V_{\Lambda_2}(\mathbf{0}) \cap \Lambda_1| = N_0$, while (c) is based on the definition of G_{Λ_1} . Now the claim follows. ■

Lemma 8. Consider two nested lattices $\Lambda_{2,0} \subset \Lambda_{1,0}$ and the scale coefficients ω_1, ω_2 such that lattices $\Lambda_2 = \omega_2 \Lambda_{2,0}$ and $\Lambda_1 = \omega_1 \Lambda_{1,0}$ are still nested. Let $N_0 \triangleq N(\Lambda_2 : \Lambda_1)$ and $\mathcal{C}_{\Lambda_2:\Lambda_1} \triangleq \cup_{\lambda_1 \in V_{\Lambda_2}(\mathbf{0}) \cap \Lambda_1} V_{\Lambda_1}(\lambda_1)$. Then

$$\lim_{\frac{\omega_2}{\omega_1} \rightarrow \infty} G(\mathcal{C}_{\Lambda_2:\Lambda_1}) = G_{\Lambda_2}.$$

Proof: Since the lattices Λ_2 and Λ_1 are scaled by different scale factors, the value $G(\mathcal{C}_{\Lambda_2:\Lambda_1})$ is not constant. On the other hand, G_{Λ_2} is constant. Notice further that the set $\mathcal{C}_{\Lambda_2:\Lambda_1}$ is a fundamental region of the lattice Λ_2 , thus its volume equals ν_{Λ_2} . Then the following holds

$$G(\mathcal{C}_{\Lambda_2:\Lambda_1}) - G_{\Lambda_2} = \frac{1}{n\nu_{\Lambda_2}^{1+\frac{2}{n}}} \left(\int_{\mathcal{C}_{\Lambda_2:\Lambda_1}} \|x^n\|^2 dx^n - \int_{V_{\Lambda_2}(\mathbf{0})} \|x^n\|^2 dx^n \right).$$

For simplicity, let us denote $\mathcal{A} = \mathcal{C}_{\Lambda_2:\Lambda_1}$, $\mathcal{B} = V_{\Lambda_2}(\mathbf{0})$ and $\Delta\nu = \nu(\mathcal{A}) - \nu(\mathcal{A} \cap \mathcal{B})$. Since $\nu(\mathcal{A}) = \nu(\mathcal{B})$, it follows that $\Delta\nu = \nu(\mathcal{B}) - \nu(\mathcal{A} \cap \mathcal{B})$. Then we obtain that

$$\begin{aligned} |G(\mathcal{A}) - G(\mathcal{B})| &= \frac{1}{n\nu_{\Lambda_2}^{1+\frac{2}{n}}} \left| \int_{\mathcal{A} \setminus \mathcal{A} \cap \mathcal{B}} \|x^n\|^2 dx^n \right. \\ &\quad \left. - \int_{\mathcal{B} \setminus \mathcal{A} \cap \mathcal{B}} \|x^n\|^2 dx^n \right| \\ &\leq \frac{1}{n\nu_{\Lambda_2}^{1+\frac{2}{n}}} (\bar{r}(\mathcal{A})^2 \Delta\nu + \bar{r}(\mathcal{B})^2 \Delta\nu) \\ &\leq \frac{\Delta\nu}{n\nu_{\Lambda_2}^{1+\frac{2}{n}}} ((\bar{r}_{\Lambda_2} + \bar{r}_{\Lambda_1})^2 + \bar{r}_{\Lambda_2}^2) \\ &\leq \frac{5\bar{r}_{\Lambda_2}^2 \Delta\nu}{n\nu_{\Lambda_2}^{1+\frac{2}{n}}} = \frac{5\omega_2^2 \bar{r}_{\Lambda_{2,0}}^2 \Delta\nu}{n\omega_2^2 \nu_{\Lambda_{2,0}}^{\frac{2}{n}} \nu_{\Lambda_2}} = \frac{5\bar{r}_{\Lambda_{2,0}}^2 \Delta\nu}{4n\nu_{\Lambda_{2,0}}^{\frac{2}{n}} \nu_{\Lambda_2}}. \end{aligned}$$

According to the above relations, in order to prove the claim of the lemma, it is sufficient to show that $\lim_{\frac{\omega_2}{\omega_1} \rightarrow \infty} \frac{\Delta\nu}{\nu_{\Lambda_2}} = 0$,

which is equivalent to

$$\lim_{\frac{\omega_2}{\omega_1} \rightarrow \infty} \frac{\nu(\mathcal{A} \cap \mathcal{B})}{\nu_{\Lambda_2}} = 1. \quad (110)$$

It is easy to see that, for any point $x^n \in V_{\Lambda_2}(\mathbf{0})$ which is at a distance larger than \bar{r}_{Λ_1} from the boundary of $V_{\Lambda_2}(\mathbf{0})$, we have $Q_{\Lambda_1}(x^n) \in V_{\Lambda_2}(\mathbf{0})$, thus $x^n \in \mathcal{A}$. This observation implies that the interior of the set $\gamma V_{\Lambda_2}(\mathbf{0})$ is included in $\mathcal{A} \cap \mathcal{B}$, where

$$\gamma = 1 - \frac{\bar{r}_{\Lambda_1}}{r_{\Lambda_2}} = 1 - \frac{\omega_1 \bar{r}_{\Lambda_{1,0}}}{\omega_2 r_{\Lambda_{2,0}}}.$$

Then we have $\gamma^n \leq \frac{\nu(\mathcal{A} \cap \mathcal{B})}{\nu_{\Lambda_2}} \leq 1$, which implies that (110) holds. With this the proof is completed. ■

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