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On the Design of Symmetric Entropy-constrained Multiple Description Scalar Quantizer with Linear Joint Decoders

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***Abstract*—This work addresses the design of symmetric entropy-constrained multiple description scalar quantizers (EC-MDSQ) with linear joint decoders (EC-LMDSQ), i.e., where some of the decoders compute the reconstruction by averaging the reconstructions of individual descriptions. Thus, the use of linear decoders reduces the space complexity at the decoder since only a subset of the codebooks need to be stored.**

The proposed design algorithm locally minimizes the Lagrangian, which is a weighted sum of the expected distortion and of the side quantizers' rates. The algorithm is inspired by the EC-MDSQ design algorithm proposed by Vaishampayan and Domaszewicz, and it is adapted from two to K descriptions. Differently from the aforementioned work, the optimization of the reconstruction values can no longer be performed separately at the decoder optimization step. Interestingly, we show that the problem is a convex quadratic optimization problem, which can be efficiently solved. Moreover, the generalization of the encoder optimization step from two to K descriptions increases drastically the amount of computations. We show how to exploit the special form of the cost function conferred by the linear joint decoders to significantly reduce the time complexity at this step.

We compare the performance of the proposed design with multiple description lattice vector quantizers (MDLVQ) and with the multiple description scheme based on successive refinement and unequal erasure protection (UEP). Our experiments show that the proposed approach outperforms MDLVQ with dimension 1 quantization, as expected. Additionally, when more codebooks are added our scheme even beats MDLVQ with quantization dimension approaching ∞, for rates sufficiently high. Furthermore, the proposed approach is also superior to UEP with dimension 1 quantization when the rates are low.

***Index Terms*—Multiple descriptions, scalar quantization, entropy-constrained, linear decoder, convex quadratic problem.**

I. Introduction

A multiple description code (MDC) generates separate descriptions of a signal, which can be individually decoded. Additionally, each subset of descriptions can be jointly decoded, improving the reconstruction as more descriptions are added to the subset. This way MDC enables a graceful degradation of performance in the case when not all descriptions arrive at the decoder. MDC has applications in the transmission of delay-sensitive data (such as video or audio/speech) over packet lossy networks, in heterogeneous multicast, in distributed data storage and diversity communication systems. Additionally, very recently, a new application of MDC emerged, namely, in the transmission over the multicast cognitive interference channel [1]. Some tutorial papers on MDC and its practical applications are [2]–[4]. A variety of practical MDC schemes have been proposed in the literature based on scalar quantizers [5]–[10], vector quantizers [11], correlated transforms [12], [13], domain partitioning [14]–[16], successively refinable codes with unequal erasure protection (UEP) [17]–[19], lattice-based vector quantizers (MDLVQ's) [20]–[26], low density matrix codes [27], [28], polar codes [29], etc. A theoretical comparison at high rates, of several representative practical MDC schemes is performed in [30].

An MDC with K descriptions consists of K encoders $f_1, \cdots, f_K$ (also referred to as *side encoders*), and 2^K-1 decoders, each decoder corresponding to a non-empty subset of descriptions [2]. We use the term *central decoder* to describe the decoder corresponding to the whole set of descriptions $\mathcal{K}$, and use the term *side decoders* to describe the decoders corresponding to the sets which have only one description. The decoders other than the side decoders are referred to as *joint decoders*[1]. Note that the central decoder is a special case of a joint decoder. In this work we are interested in symmetric MDC's, where the attribute "symmetric" refers to the fact that all descriptions have the same rate, while the quality of the reconstruction at joint decoders depends on the number rather than on the particular set of jointly decoded descriptions.

As pointed out earlier, an MDC scheme generating K descriptions has 2^K-1 decoders. If each decoder needs to store some information to enable decoding, then the storage space requirement increases significantly with the number K of descriptions. High storage space needs could be an issue, especially in applications where the memory resources are scarce (for example, transmission to mobile devices). Such applications motivate the study of MDC with reduced storage space decoder.

Among existing symmetric MDC schemes with reduced storage space decoders, the most popular are UEP and MDLVQ. The UEP scheme employs a successively refinable source code (SRC) in conjunction with maximum distance separable channel codes, such as Reed-Solomon (RS) codes, of various strengths. The output of the SRC is partitioned into consecutive segments of non-decreasing lengths. These segments are protected against erasures using RS codes of the same length K, but non-increasing strengths. Finally, the

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[1]The terms "side encoder" and "joint decoder" are borrowed from [10].

K descriptions are formed across the channel codewords. When only k out of K descriptions are received, all RS codewords protected against at most $K-k$ erasures can be decoded correctly, ensuring that a prefix of the SRC can be successfully recovered and further decoded. The application of the UEP scheme in image/video transmission over packet lossy channels was extensively studied [17]–[19].

The MDLVQ scheme is based on vector quantization with lattice codebooks. More specifically, it uses a fine lattice (*central lattice*) as the codebook of the central decoder, and a coarse lattice (*side lattice*) as the codebook for the side decoders. The MDLVQ scheme was introduced in [20] for the case of two symmetric descriptions and generalized to larger K in [21]. In the latter case joint decoders corresponding to at least two, but less than K descriptions, use the average of the lattice points corresponding to received descriptions as reconstruction. Further study of MDLVQ was performed in [22]–[24], [26]. In [25] an MDLVQ of dimension 1 is proposed with the side codebooks being shifts of the side lattice.

In this paper we propose a technique for the design of symmetric entropy-constrained multiple description scalar quantizer with reduced storage space decoder. A multiple description scalar quantizer (MDSQ) is an MDC where each side encoder is a scalar quantizer. An entropy-constrained MDSQ (EC-MDSQ) is an MDSQ where each side encoder is an entropy-constrained quantizer. The MDSQ was introduced in [5] for the case of two descriptions, where a design algorithm was also proposed. The design of EC-MDSQ for the case of two descriptions was addressed in [6]. Further work on MDSQ or EC-MDSQ design for two descriptions was performed in [7]–[9], while the design of MDSQ with K descriptions was considered in [9], [10]. Additionally, the construction of multiple description vector quantizers was addressed in [11].

The traditional MDSQ and EC-MDSQ store a codebook for each decoder. The solution that we propose is to store a codebook only for each side decoder and a few joint decoders, and generate the reconstructions for the other joint decoders as the average of the received descriptions. The latter decoders will be referred to as linear decoders. We use the acronym EC-LMDSQ (with "L" for linear) for the proposed EC-MDSQ system. Our EC-LMDSQ scheme is inspired by the existing MDLVQ framework for dimension 1, but it is different from it in two ways: 1) we seek to optimize the central and side codebooks instead of using lattice-based codebooks; 2) we include the option that more joint decoders store optimized codebooks.

The algorithm we propose for EC-LMDSQ design is related to the EC-MDSQ design algorithm introduced by Vaishampayan and Domaszewicz for the case of two descriptions [6]. The algorithm locally minimizes the Lagrangian formed as a weighted sum of the expected distortion and of the side encoders' rates. However, there are significant differences. One difference resides in the decoder optimization step, which becomes more complex since the codebooks can no longer be optimized separately as in the traditional case. Fortunately, it turns out that the decoder optimization problem is a convex quadratic optimization problem, which can be solved efficiently using well established techniques [31]. As for the encoder optimization step, the direct extension from two to K descriptions would lead to an escalation of the time complexity. To address this problem we take advantage of the particular form of the cost function due to the linear joint decoders, and drastically reduce the amount of operations performed at this step.

We compare experimentally the performance of the proposed scheme with MDLVQ and UEP. Our results show that EC-LMDSQ outperforms MDLVQ of dimension 1 for the whole range of rates tested, as expected. On the other hand, when more codebooks are added and the rate is sufficiently high, the proposed scheme becomes superior even than MDLVQ with quantization dimension approaching ∞. Moreover, for low rates EC-LMDSQ is also better than UEP with dimension 1 quantization.

The paper is structured as follows. The following section formally defines the MDSQ and EC-MDSQ and discusses the problem of optimal MDSQ/EC-MDSQ design. Section III introduces the EC-LMDSQ. Section IV presents an overview of the proposed EC-LMDSQ design algorithm. Further, the details of the decoder, respectively encoder, optimization step of the latter algorithm are addressed in section V, respectively VI. Extensive experimental results and the comparison with MDLVQ and UEP are reported in section VII. Finally, section VIII concludes the paper.

II. Definitions and Notations

The encoder of an MDSQ generating K descriptions consists of a K-tuple of side encoders $\mathbf{f} = (f_1, \cdots, f_K)$, where $f_k : \mathbb{R} \to \{1, \cdots, M_k\}$, for some positive integer M_k, $1 \le k \le K$. Given a source sample x, the encoder maps x into $(i_1, \cdots, i_K) \in \mathcal{I}_{\mathcal{K}}$, where $i_k = f_k(x)$, $1 \le k \le K$, and $\mathcal{I}_{\mathcal{K}}$ denotes the set of all K-tuples generated by the encoder. For each $(i_1, \cdots, i_K) \in \mathcal{I}_{\mathcal{K}}$, let $A_{i_1,\cdots,i_K} \triangleq \{x | f_k(x) = i_k, 1 \le k \le K\}$. Then $\{A_{i_1,\cdots,i_K} | (i_1, \cdots, i_K) \in \mathcal{I}_{\mathcal{K}}\}$ is a partition of $\mathbb{R}$, referred to as the *central partition*. We will assume that the central partition consists of convex cells, i.e., intervals, and denote by N the number of these intervals[2]. Another important notion related to the MDSQ is the index assignment (IA), defined as the mapping $h : \{1, 2, \cdots, N\} \to \mathcal{I}_{\mathcal{K}}$, where $h(\ell) = (i_1, \cdots, i_K)$ if the ℓ-th interval in the central partition equals the set $A_{i_1,\cdots,i_K}$. Then it is clear that the encoder $\mathbf{f}$ is completely specified by the central partition and the IA. Additionally, notice that the mapping f_k can be regarded as the encoder of a quantizer, which is referred to as the kth *side quantizer*.

The index i_k is transmitted over the kth channel, $1 \le k \le K$. Each channel has some probability of breaking down. Thus, at the receiver there are two kinds of situations with respect to each description: either the channel works properly and the received index is correct, or the channel breaks down and nothing is received. Let $\mathcal{L} = \{l_1, \cdots, l_s\} \subseteq \mathcal{K}$ denote a subset of descriptions, $1 \le s \le K$, where $\mathcal{K} \triangleq \{1, \cdots, K\}$. Assume that only the descriptions in subset $\mathcal{L}$ are received at

[2] As argued in [5], [9] the constraint of cell convexity in the central partition does not preclude optimality for MDSQ's achieving points on the lower convex hull of the set of achievable tuples of distortions and rates.

the decoder. Then the decoder used to reconstruct the source sample is denoted by $g_{\mathcal{L}}$ and it maps each s-tuple $(i_{l_1}, \cdots, i_{l_s})$ to some reconstruction value denoted by $a^{(\mathcal{L})}_{i_{l_1},\cdots,i_{l_s}} \in \mathbb{R}$, where $1 \le i_{l_k} \le M_k$, for all $1 \le k \le s$. Denote by $\mathbf{g}$, the $(2^K - 1)$-tuple of decoders $(g_{\mathcal{L}})_{\mathcal{L}\subseteq\mathcal{K}}$.

Let us evaluate now the expected distortion of the source reconstruction when the subset $\mathcal{L} = \{l_1, \cdots, l_s\}$ of descriptions are received at the decoder. For this let us denote by $f_X(\cdot)$ the probability density function (pdf) of the source and let $\mathcal{I}_{\mathcal{L}}$ denote the set of all possible s-tuples of indexes corresponding to $\mathcal{L}$. More specifically, $\mathcal{I}_{\mathcal{L}} \triangleq \{(i_{l_1}, \cdots, i_{l_s})|$ there is $x \in \mathbb{R}$, such that $f_{l_k}(x) = i_{l_k}$, for all $1 \le k \le s\}$. Additionally, for each $k, 1 \le k \le s$, and $i_{l_k}, 1 \le i_{l_k} \le M_k$, let $f^{-1}_{l_k}(i_{l_k})$ denote the set of values which are mapped by f_{l_k} to the index i_{l_k}. Then for each s-tuple $(i_{l_1}, \cdots, i_{l_s}) \in \mathcal{I}_{\mathcal{L}}$, the set of all values x mapped by the encoders to this s-tuple is $A^{(\mathcal{L})}_{i_{l_1},\cdots,i_{l_s}} \triangleq \bigcap_{k=1}^{s} f^{-1}_{l_k}(i_{l_k})$. It is clear that the collection of sets $A^{(\mathcal{L})}_{i_{l_1},\cdots,i_{l_s}}$ with $(i_{l_1}, \cdots, i_{l_s}) \in \mathcal{I}_{\mathcal{L}}$, forms a partition of $\mathbb{R}$. Further, using the squared error as the distortion measure we obtain the expected distortion at the decoder corresponding to subset $\mathcal{L}$ as follows,

$$D_{\mathcal{L}}(\mathbf{f}, \mathbf{g}) = \sum_{(i_{l_1},\cdots,i_{l_s})\in\mathcal{I}_{\mathcal{L}}} \int_{A^{(\mathcal{L})}_{i_{l_1},\cdots,i_{l_s}}} (x - a^{(\mathcal{L})}_{i_{l_1},\cdots,i_{l_s}})^2 f_X(x) dx. \quad (1)$$

Let $R_k(\mathbf{f})$ denote the rate for side encoder $k, 1 \le k \le K$. We do not include the decoder $\mathbf{g}$ in the notation since the rate does not depend on it. In a fixed-rate MDSQ all indexes of a side encoder are encoded using the same number of bits. Therefore, one has $R_k(\mathbf{f}) = \lceil \log_2 M_k \rceil$ for all k. In an EC-MDSQ the indexes generated by a side encoder are first compressed using an entropy coder and only after that the resulted codestream is transmitted over the corresponding channel. Like in most past work on the design of entropy-constrained quantizer systems, we assume that the rate of each side encoder equals the entropy of the corresponding side quantizer [6], [9], [11]. Therefore, we have for all $1 \le k \le K$,

$$R_k(\mathbf{f}) = -\sum_{n=1}^{M_k} P^{(k)}_n \log_2 P^{(k)}_n, \quad (2)$$

where $P^{(k)}_n$ denotes the probability of the nth cell of the kth side quantizer, i.e., the set $f^{-1}_k(n)$. The problem of optimal MDSQ design, be it fixed-rate or entropy-constrained, can be formulated as the minimization of the distortion at the central decoder with constraints imposed on the rates and on the distortions at the other decoders, i.e.,

$$\begin{aligned} \text{minimize}_{\ \mathbf{f},\mathbf{g}} \quad & D_{\mathcal{K}}(\mathbf{f}, \mathbf{g}) \\ \text{subject to} \quad & R_k(\mathbf{f}) \le R_k, \quad k = 1, 2, \cdots, K, \\ & D_{\mathcal{L}}(\mathbf{f}, \mathbf{g}) \le D_{\mathcal{L}}, \quad \mathcal{L} \subset \mathcal{K}. \end{aligned} \quad (3)$$

This formulation was adopted in [5], [6]. Other authors considered the criterion of minimizing the overall expected distortion $\bar{D}(\mathbf{f}, \mathbf{g})$ with constraints on the rates [7], [8], [10], i.e.,

$$\begin{aligned} \text{minimize}_{\ \mathbf{f},\mathbf{g}} \quad & \bar{D}(\mathbf{f}, \mathbf{g}) = \sum_{\mathcal{L}\subseteq\mathcal{K}} p_{\mathcal{L}} D_{\mathcal{L}}(\mathbf{f}, \mathbf{g}) \\ \text{subject to} \quad & R_k(\mathbf{f}) \le R_k, \quad k = 1, 2, \cdots, K, \end{aligned} \quad (4)$$

where $p_{\mathcal{L}}$ is the probability that only the descriptions in subset $\mathcal{L}$ are received.

In the case of fixed-rate MDSQ the constraints on the rates can be easily handled by imposing the necessary number of side quantizer cells, i.e., by choosing the appropriate value for each M_k. On the other hand, the distortion constraints in problem (3), as well as the rate constraints in the case of EC-MDSQ are handled in prior work by means of Lagrangian relaxation, i.e., they are incorporated in the Lagrangian cost function. Thus, the problem is converted to the unconstrained problem of minimizing the Lagrangian. The latter problem can find the solutions to problem (3), respectively (4), which are situated on the lower convex hull of the set of achievable tuples of distortions and rates. For the aforementioned reason, the authors of [9], [11] directly formulate the optimization problem as the minimization of the weighted sum of the distortions and rates.

Thus, for both problems (3) and (4), when the encoder is fixed and there are no constraints imposed on the reconstruction values, the decoder can be optimized by separately minimizing each term in the summation in (1), for each $\mathcal{L} \subset \mathcal{K}$. This turns into a simpler problem of separately minimizing each integral, leading to the solution:

$$a^{(\mathcal{L})}_{i_{l_1},\cdots,i_{l_s}} = E\left[X | X \in A^{(\mathcal{L})}_{i_{l_1},\cdots,i_{l_s}}\right] = \frac{\int_{A^{(\mathcal{L})}_{i_{l_1},\cdots,i_{l_s}}} x f_X(x) dx}{\int_{A^{(\mathcal{L})}_{i_{l_1},\cdots,i_{l_s}}} f_X(x) dx}, \quad (5)$$

for all $\mathcal{L} \subseteq \mathcal{K}$ and $(i_{l_1}, \cdots, i_{l_s}) \in \mathcal{I}_{\mathcal{L}}$.

III. Definition and Motivation of EC-LMDSQ

Notice that an MDSQ or EC-MDSQ with K descriptions has $2^K - 1$ decoders, each with its own codebook. Moreover, the size of each codebook increases exponentially with the number of corresponding descriptions. For example, if R is the rate of each side encoder, then the size of the codebook of decoder $g_{\mathcal{L}}$ for some subset $\mathcal{L}$ of s descriptions, $1 \le s \le K$, could reach the value 2^{sR}. Thus, the total number of reconstruction values for decoder $\mathbf{g}$ could amount to $\sum_{s=1}^{K} \binom{K}{s} (2^R)^s = (1 + 2^R)^K - 1$. As the number K of descriptions increases, storing all these values becomes an issue, especially in applications where the memory resources are scarce (for example, mobile devices).

To address this problem we propose to store the codebook only for each side decoder and a few joint decoders, and generate the other codebooks as linear combinations of the reconstruction values from the side codebooks. Since we are interested in the case of symmetric descriptions we will consider equal weights for all descriptions. More specifically, if $\mathcal{L} = \{l_1, \cdots, l_s\}$ is a subset of descriptions for which the codebook is not stored, then its reconstruction values are computed according to

$$a^{(\mathcal{L})}_{i_{l_1},\cdots,i_{l_s}} = \frac{1}{s}\sum_{k=1}^{s} a^{(l_k)}_{i_{l_k}}, \quad (6)$$

for any $(i_{l_1},\cdots,i_{l_s}) \in \mathcal{I}_{\mathcal{L}}$. In order to increase the performance, the stored codebooks should correspond to the subsets of descriptions with the highest probabilities of being received. In the case of symmetric, independent channels with probability of channel failure $q < 0.5$, the probability of receiving a particular set of descriptions increases with the set size. Therefore, we will store the codebooks of all subsets of at least K_0+1 descriptions, for some fixed value $K_0, 1 \leq K_0 \leq K$, and will use the linear rule (6) to generate the codebooks for subsets of 2 up to K_0 descriptions. We will use the acronym EC-LMDSQ for such an EC-MDSQ system. Notice that the choice of the value K_0 should be done such that to strike a balance between the codebook storage needs and the LMDSQ performance.

We are interested in the design of optimal symmetric EC-LMDSQ, where the attribute "symmetric" refers to the fact that the rates of all descriptions are the same, i.e., $R_1(\mathbf{f}) = \cdots = R_K(\mathbf{f})$, and $p_{\mathcal{L}}$ depends only on the number of received descriptions $\mathcal{L}$, i.e., $p_{\mathcal{L}} = p(|\mathcal{L}|)$. For simplicity we will use the term EC-LMDSQ instead of "symmetric EC-LMDSQ" in the rest of the paper.

It is interesting to compare the complexity of the proposed scheme with existing MDC schemes with reduced storage space at the decoder, such as UEP and MDLVQ. The UEP scheme has higher encoding and decoding time complexity due to the use of erasure codes. In particular, the encoding and decoding time complexity per sample is $O(K^2)$ versus $O(K)$ for EC-LMDSQ. On the other hand, MDLVQ uses the same average decoding rule, but the side codebooks are not optimized, while our proposed scheme has optimized reconstruction values. Thus, EC-LMDSQ has a higher space complexity than MDLVQ since the latter do not need to store any codebook. On the other hand, we expect the performance of EC-LMDSQ to be higher than MDLVQ of dimension 1 because of the optimization of the stored codebooks. Our experimental results show that indeed this is the case.

IV. Overview of EC-LMDSQ Design Algorithm

As mentioned in the previous section, since we are concerned with the symmetric case, for any $0 \leq s \leq K$, and any set $\mathcal{L}$ of s descriptions, the probability of receiving only the set $\mathcal{L}$ will be denoted from now on by $p(s)$. For example, when the descriptions are transmitted over independent channels with the same failure rate q, we have $p(s) = (1-q)^s q^{K-s}$. Thus, the expected distortion of the EC-LMDSQ can be rewritten as

$$\bar{D}(\mathbf{f},\mathbf{g}) = \sum_{s=1}^{K} p(s) \times \sum_{\mathcal{L}=\{l_1,\cdots,l_s\}\subseteq\mathcal{K}} D_{\mathcal{L}}(\mathbf{f},\mathbf{g}).$$

We consider the optimization problem

$$\begin{aligned} \text{minimize}_{\mathbf{f},\mathbf{g}} \quad & \bar{D}(\mathbf{f},\mathbf{g}) \\ \text{subject to} \quad & R_k(\mathbf{f}) \leq R, \quad k = 1,2,\cdots,K, \end{aligned} \quad (7)$$

for some target rate R. The Lagrangian for this problem is

$$\mathcal{L}(\mathbf{f},\mathbf{g},\boldsymbol{\mu}) = \bar{D}(\mathbf{f},\mathbf{g}) + \sum_{k=1}^{K} \mu_k R_k(\mathbf{f}),$$

where $\boldsymbol{\mu} = (\mu_1,\cdots,\mu_K)$ is a K-tuple of positive Lagrangian multipliers. Consider now the problem of minimizing the Lagrangian for a given $\boldsymbol{\mu}$, i.e.,

$$\text{minimize}_{\mathbf{f},\mathbf{g}} \quad \mathcal{L}(\mathbf{f},\mathbf{g},\boldsymbol{\mu}). \quad (8)$$

According to [32] any solution $(\mathbf{f}^*,\mathbf{g}^*)$ to problem (8) is also a solution to the problem

$$\begin{aligned} \text{minimize}_{\mathbf{f},\mathbf{g}} \quad & \bar{D}(\mathbf{f},\mathbf{g}) \\ \text{subject to} \quad & R_k(\mathbf{f}) \leq R_k(\mathbf{f}^*), \quad k = 1,2,\cdots,K. \end{aligned}$$

Conversely, any solution of problem (7) which lies on the lower convex hull of the set of all possible $(K+1)$-tuples $(\bar{D}(\mathbf{f},\mathbf{g}), R_1(\mathbf{f}),\cdots,R_k(\mathbf{f}))$, can be found by solving problem (8) for some K-tuple $\boldsymbol{\mu}$.

Using relation (2), we obtain that

$$\mathcal{L}(\mathbf{f},\mathbf{g},\boldsymbol{\mu}) = \bar{D}(\mathbf{f},\mathbf{g}) + \sum_{k=1}^{K}\mu_k \sum_{i_k=1}^{M_k} P^{(k)}_{i_k}(\mathbf{f}) \log_2 \frac{1}{P^{(k)}_{i_k}(\mathbf{f})},$$

where we use $P^{(k)}_{i_k}(\mathbf{f})$ instead of $P^{(k)}_{i_k}$ in order to emphasize that $P^{(k)}_{i_k}(\mathbf{f})$ is a function of the encoder $\mathbf{f}$. As in prior work on entropy-constrained quantizer systems design [6], [41], in order to solve problem (8) we minimize a more general cost function which has the same minimum value as $\mathcal{L}(\mathbf{f},\mathbf{g},\boldsymbol{\mu})$. To this end consider the auxiliary variables $Q^{(k)}_{i_k}$ with positive values, for $1 \leq i_k \leq M_k$, $1 \leq k \leq K$, and the cost

$$\mathcal{C}(\mathbf{f},\mathbf{g},\boldsymbol{\mu},\mathbf{Q}) = \bar{D}(\mathbf{f},\mathbf{g}) + \sum_{k=1}^{K}\mu_k \sum_{i_k=1}^{M_k} P^{(k)}_{i_k}(\mathbf{f}) \log_2 \frac{1}{Q^{(k)}_{i_k}}, \quad (9)$$

where $\mathbf{Q}$ denotes the vector formed with the $\sum_{k=1}^{K} M_k$ variables $Q^{(k)}_{i_k}$. Consider now the more general problem

$$\begin{aligned} \text{minimize}_{\mathbf{f},\mathbf{g},\mathbf{Q}} \quad & \mathcal{C}(\mathbf{f},\mathbf{g},\boldsymbol{\mu},\mathbf{Q}) \\ \text{subject to} \quad & Q^{(k)}_{i_k} \geq 0, \quad 1 \leq k \leq K, \quad 1 \leq i_k \leq M_k, \\ & \textstyle\sum_{i_k=1}^{M_k} Q^{(k)}_{i_k} \leq 1, \quad 1 \leq k \leq K. \end{aligned} \quad (10)$$

It is known that the optimal value of problem (10) is achieved for $Q^{(k)}_{i_k} = P^{(k)}_{i_k}(\mathbf{f})$ for all k and i_k [33]. Thus, we can find the solution to problem (8) by solving the more general problem (10).

The algorithm to solve problem (10) is a locally optimal algorithm similar in spirit to the algorithm employed in [6] for the design of EC-MDSQ with two descriptions. Namely, the algorithm starts with an initial configuration and it proceeds in iterations. At every iteration there are three steps: 1) keep **f** and **g** fixed and optimize **Q**; 2) keep the encoder **f** and the vector **Q** fixed and optimize the decoder **g**; 3) keep **g** and **Q** fixed and optimize the encoder **f**. The first step is accomplished by letting $Q^{(k)}_{i_k} = P^{(k)}_{i_k}(\mathbf{f})$ for all k and i_k, as argued earlier. The encoder optimization step can be carried out as in prior work on various quantizer systems' design, but we will show how to leverage the linear decoders to reduce the time complexity of this step comparing to the general purpose technique. Finally, the decoder optimization step poses a challenge which is not encountered in other quantizer design problems since the codebooks for some of the decoders cannot be optimized separately. Fortunately, as we show in Section V, the problem turns out to be a convex quadratic optimization problem, for which efficient solutions are well known.

The proposed design algorithm is only guaranteed to converge to a local optimum, thus the selection of the initial encoder is very important. Recall that an encoder is fully represented by the intervals in the central partition and the IA. The problem of optimal IA design was addressed in [5], [7], [10], [20]–[22], [25], [34], [35]. It is worth pointing out that the optimality of the IA's proposed in the literature holds under restrictive assumptions. Specifically, Vaishampayan [5] proposes good IAs for the case $K = 2$ based on arguments that hold at high rates. The IA's proposed for MDLVQ's in [20]–[22], [25] are optimal under the assumption of high rate and high reuse index. Note that the reuse index represents how many times each index of a description is assigned a central lattice point. In [7], [10] it is shown that the staggered IA is optimal for symmetric MDSQ with convex cells and the result holds for any K and any pdf.

We would like to point out that locally optimal algorithms are common practice in the design of various quantizer systems. Interestingly, there are also cases when they ensure the global optimality of the solution. This happens when the local minimum is unique. Sufficient conditions for the uniqueness of the local minimum in the case of fixed-rate scalar quantizer, were given by Trushkin in [36]. Similar conditions were shown to be sufficient for the uniqueness of the local minimum in the case of fixed-rate MDSQ's with convex cells, with respect to a fixed IA, by Dumitrescu and Wu in [10]. In particular, these conditions are satisfied for log-concave density functions and the squared error distortion measure. Moreover, the recent work [37] establishes sufficient conditions for the uniqueness of the local minimum in the case of fixed-rate noisy scalar quantizers with random index assignment. On the other hand, globally optimal algorithms for MDSQ design were proposed for discrete distributions, under the assumption that all cells are convex in [7]–[9].

V. Decoder Optimization Step

At this step the central partition and the IA are fixed, and the codebook is optimized. In other words, the goal at this step is to choose the reconstruction values for the side codebooks and for the joint codebooks corresponding to at least $K_0 + 1$ descriptions, such that the expected distortion to be minimized. As it can be seen from equation (9) the decoder $\mathbf{g}$ only affects the expected distortion. Thus, the decoder $\mathbf{g}$ should be chosen such that to minimize $\bar{D}(\mathbf{f}, \mathbf{g})$. To this end, it is useful to write the expected distortion as $\bar{D}(\mathbf{f}, \mathbf{g}) = \bar{D}_1(\mathbf{f}, \mathbf{g}) + \bar{D}_2(\mathbf{f}, \mathbf{g}) + p(0)\sigma^2$, where σ^2 is the variance of the source, $\bar{D}_1(\mathbf{f}, \mathbf{g})$ is the contribution to the expected distortion when at least one and at most K_0 descriptions are received, while $\bar{D}_2(\mathbf{f}, \mathbf{g})$ is the contribution when at least $K_0 + 1$ descriptions are available at the decoder. Using (6) we obtain that

$$\bar{D}_1(\mathbf{f}, \mathbf{g}) = \sum_{s=1}^{K_0} p(s) \times \sum_{\mathcal{L}=\{l_1,\cdots,l_s\}\subseteq\mathcal{K}} \sum_{(i_{l_1},\cdots,i_{l_s})\in\mathcal{I}_\mathcal{L}} \int_{A^{(\mathcal{L})}_{i_{l_1},\cdots,i_{l_s}}} \left(x - \frac{1}{s}\sum_{k=1}^{s} a^{(l_k)}_{i_{l_k}}\right)^2 f_X(x)dx, \tag{11}$$

$$\bar{D}_2(\mathbf{f}, \mathbf{g}) = \sum_{s=K_0+1}^{K} p(s) \times \sum_{\mathcal{L}=\{l_1,\cdots,l_s\}\subseteq\mathcal{K}} \sum_{(i_{l_1},\cdots,i_{l_s})\in\mathcal{I}_\mathcal{L}} \int_{A^{(\mathcal{L})}_{i_{l_1},\cdots,i_{l_s}}} \left(x - a^{(\mathcal{L})}_{i_{l_1},\cdots,i_{l_s}}\right)^2 f_X(x)dx.$$

It is clear that $\bar{D}_1(\mathbf{f}, \mathbf{g})$ and $\bar{D}_2(\mathbf{f}, \mathbf{g})$ can be optimized separately. Moreover, $\bar{D}_2(\mathbf{f}, \mathbf{g})$ can be optimized by independently minimizing each integral in the summation, by applying equation (5) for all $\mathcal{L} \subseteq \mathcal{K}$ with $s = |\mathcal{L}| \geq K_0 + 1$, and $(i_{l_1}, \cdots, i_{l_s}) \in \mathcal{I}_\mathcal{L}$.

We are left now with the task of finding the side codebooks which minimize $\bar{D}_1(\mathbf{f}, \mathbf{g})$. Although the integrals in (11) can no longer be minimized independently, the minimization of $\bar{D}_1(\mathbf{f}, \mathbf{g})$ can still be handled easily since the problem is a convex optimization problem. The convexity of $\bar{D}_1(\mathbf{f}, \mathbf{g})$ follows from the fact that the squaring function is convex, while composition with an affine mapping, multiplication with a nonnegative constant, addition and integration are all operations that preserve convexity. Actually, the problem is a convex quadratic optimization problem. In other words, if $\mathbf{y}$ denotes the $(\sum_{k=1}^{K} M_k)$-dimensional vector which consists of all reconstruction values of the side codebooks:

$$\mathbf{y} = (a^{(1)}_1, a^{(1)}_2, \cdots, a^{(1)}_{M_1}, \cdots, a^{(j)}_1, \cdots, a^{(j)}_{i_j}, \cdots, a^{(j)}_{M_j}, \cdots, a^{(K)}_1, \cdots, a^{(K)}_{M_K})^T, \tag{12}$$

then $\bar{D}_1(\mathbf{f}, \mathbf{g})$ can be written as

$$\bar{D}_1(\mathbf{f}, \mathbf{g}) = \mathbf{y}^T\mathbf{B}\mathbf{y} + \mathbf{u}^T\mathbf{y} + r, \tag{13}$$

where r is a real value, $\mathbf{u}$ is a fixed $(\sum_{k=1}^{K} M_k)$-dimensional column vector and $\mathbf{B}$ is a fixed $(\sum_{k=1}^{K} M_k) \times (\sum_{k=1}^{K} M_k)$ symmetric positive semidefinite matrix. Note that $(\cdot)^T$ denotes the transpose of a vector or matrix. Then minimizing (13) reduces to solving for $\mathbf{y}$ the following set of linear equations

$$2\mathbf{B}\mathbf{y} + \mathbf{u} = \mathbf{0}, \tag{14}$$

where $\mathbf{0}$ denotes the $(\sum_{k=1}^{K} M_k)$-dimensional all zero column vector. Once the matrix $\mathbf{B}$ is computed, solving (14) takes $O(K^3M^3)$ operations, for instance by using Gaussian elimination, where $M \triangleq \max_{1\leq k\leq K} M_k$. However, in order to evaluate the complexity of the decoder optimization step we also need to account for the construction of matrix $\mathbf{B}$ and vector $\mathbf{u}$. To this end it is useful to have the explicit form of $\mathbf{B}$ and vector $\mathbf{u}$, which are derived next.

First we need to introduce some more notations. We use the notation $\mathbf{i}$ for a K-tuple $(i_1, \cdots, i_K) \in \mathcal{I}_\mathcal{K}$. Additionally, for each $\mathbf{i} \in \mathcal{I}_{\mathcal{K}}$ denote

$$c_{\mathbf{i}} = \int_{A_{\mathbf{i}}} x f_X(x) dx, \quad P_{\mathbf{i}} = \int_{A_{\mathbf{i}}} f_X(x) dx.$$

Further, for every integer $m, 1 \leq m \leq \sum_{k=1}^{K} M_k$, define $\jmath(m)$ and $w(m)$ as follows. $\jmath(m)$ is the unique integer in the range 1 to K such that $\sum_{k=1}^{\jmath(m)-1} M_k < m \leq \sum_{k=1}^{\jmath(m)} M_k$, and $w(m) \triangleq m - \sum_{k=1}^{\jmath(m)-1} M_k$. It is easy to see that the m-th component of vector $\mathbf{y}$ is $a_{w(m)}^{(\jmath(m))}$. Additionally, for every $1 \leq j \leq K$ and $1 \leq n \leq M$, denote by $C_n^{(j)}$ the sum of $c_{\mathbf{i}}$'s for all central cells $A_{\mathbf{i}}$ such that the jth component of the K-tuple $\mathbf{i}$ equals n, i.e.,

$$C_n^{(j)} = \sum_{\mathbf{i} \in \mathcal{I}_{\mathcal{K}}, i_j = n} c_{\mathbf{i}}. \tag{15}$$

Recall that for every $1 \leq j \leq K$ and $1 \leq n \leq M$, we denote by $P_n^{(j)}$ the probability of cell n of side quantizer j. Further, for every $1 \leq j, j' \leq K$ and $1 \leq n, n' \leq M$, denote by $P_{n,n'}^{(j,j')}$ ($j \neq j'$ or $n \neq n'$) the probability of the intersection of cell n of side quantizer j and cell n' of side quantizer j'. Formally,

$$P_n^{(j)} = \sum_{\mathbf{i} \in \mathcal{I}_{\mathcal{K}}, i_j = n} P_{\mathbf{i}}, \quad P_{n,n'}^{(j,j')} = \sum_{\substack{\mathbf{i} \in \mathcal{I}_{\mathcal{K}} \\ i_j = n, i_{j'} = n'}} P_{\mathbf{i}}. \tag{16}$$

Denote further by $\mathcal{U}$ the second moment of the pdf, i.e., $\mathcal{U} = \int_{\mathbb{R}} x^2 f_X(x) dx$.

Proposition 1. *Let*

$$r = \sum_{s=1}^{K_0} p(s) \times \binom{K}{s} \mathcal{U}, \tag{17}$$

and let $\mathbf{u}$ be the $(\sum_{k=1}^{K} M_k)$-dimensional vector whose m-th component, $1 \leq m \leq \sum_{k=1}^{K} M_k$, is defined as follows

$$u_m = -2 \left(\sum_{s=1}^{K_0} p(s) \frac{1}{K} \binom{K}{s} \right) C_{w(m)}^{(\jmath(m))}. \tag{18}$$

Finally, let $\mathbf{B}$ be the $(\sum_{k=1}^{K} M_k) \times (\sum_{k=1}^{K} M_k)$ symmetric matrix with elements $b_{m\ell}$, $1 \leq m, \ell \leq \sum_{k=1}^{K} M_k$, defined as follows

$$b_{m\ell} = \begin{cases} \left(\sum_{s=1}^{K_0} p(s) \binom{K}{s} \frac{1}{Ks} \right) P_{w(m)}^{(\jmath(m))}, & m = \ell \\ \left(\sum_{s=1}^{K_0} p(s) \binom{K}{s} \frac{(s-1)}{Ks(K-1)} \right) P_{w(m),w(\ell)}^{(\jmath(m),\jmath(\ell))}, & m \neq \ell. \end{cases} \tag{19}$$

Then matrix $\mathbf{B}$ is positive definite and equality (13) holds for $\mathbf{y}$ defined in (12).

Before proving this proposition let us first evaluate the time complexity of constructing the vector $\mathbf{u}$ and the matrix $\mathbf{B}$. Notice first that the factors $\sum_{s=1}^{K_0} p(s) \frac{1}{K} \binom{K}{s}$, $\sum_{s=1}^{K_0} p(s) \binom{K}{s} \frac{1}{Ks}$ and $\sum_{s=1}^{K_0} p(s) \binom{K}{s} \frac{(s-1)}{Ks(K-1)}$ can be precomputed and stored at the beginning of the algorithm. Therefore, the number of operations needed to construct $\mathbf{u}$ and $\mathbf{B}$ is proportional with the number of operations for computing all quantities $C_n^{(j)}$, $P_n^{(j)}$ and $P_{n,n'}^{(j,j')}$. The computation of the latter quantities can be divided into two stages. The first stage computes all values $c_{\mathbf{i}}$ and $P_{\mathbf{i}}$ for all $\mathbf{i} \in \mathcal{I}_{\mathcal{K}}$. The computation of the integrals in the expression of $c_{\mathbf{i}}$ and $P_{\mathbf{i}}$ can be handled by dividing the interval $A_{\mathbf{i}}$ into small subintervals, say of size δ, and applying the trapezoidal rule on each small interval. Assuming that the pdf has a finite support included in some interval $\mathcal{B}$ of size B, the total running time amounts to $O(B/\delta)$. Alternatively, we can use a preprocessing step, where the cumulative zero-th and first order moments are computed and stored for the discretization of the pdf obtained by dividing $\mathcal{B}$ into subintervals of size δ. Then each $P_{\mathbf{i}}$, respectively $c_{\mathbf{i}}$, can be computed in $O(1)$ time as the difference of two cumulative zero-th, respectively first, order moments. This way the running time of the first stage becomes $O(N)$, while the preprocessing step requires $O(B/\delta)$ operations.

The second stage computes all $C_n^{(j)}$, $P_n^{(j)}$ and $P_{n,n'}^{(j,j')}$ based on equations (15) and (16). Note that for each j, the computation of all values $C_n^{(j)}$, $P_n^{(j)}$, for $1 \leq n \leq M$, takes $O(N)$ time. This is because every central cell $A_{\mathbf{i}}$ is included in exactly one cell of side quantizer j. Over all j's this amounts to $O(KN)$ operations. Likewise, computing all values $P_{n,n'}^{(j,j')}$ for a fixed pair (j, j') and all pairs n, n', takes $O(N + M^2)$ operations. Note that the term M^2 was included to account for assigning the value 0 to those $P_{n,n'}^{(j,j')}$ for which the sum in (16) does not have any term. Over all pairs (j, j') this amounts to $O(K^2(N + M^2))$ operations. Consequently, the time complexity of the second stage equals $O(K^2(N+M^2))$.

Proof of Proposition 1: In order to prove this claim we first expand each integral in (11) as the summation of integrals over cells in the central partition. More specifically, notice that for each $\mathcal{L}$ and each $(i_{l_1}, \cdots, i_{l_s}) \in \mathcal{I}_{\mathcal{L}}$, the set $A_{i_{l_1}, \cdots, i_{l_s}}^{(\mathcal{L})}$ is the union of all central partition cells $A_{\mathbf{j}}$, with $\mathbf{j} = (j_1, \cdots j_K) \in \mathcal{I}_{\mathcal{K}}$ and $j_{l_1} = i_{l_1}, \cdots, j_{l_s} = i_{l_s}$. Then the integral over $A_{i_{l_1}, \cdots, i_{l_s}}^{(\mathcal{L})}$ can be written as the sum of integrals over each constituent $A_{\mathbf{j}}$. Furthermore, notice that each cell $A_{\mathbf{j}}$ of the central partition appears in exactly one set $A_{i_{l_1}, \cdots, i_{l_s}}^{(\mathcal{L})}$. Thus, one obtains that

$$\bar{D}_1(\mathbf{f}, \mathbf{g}) = \sum_{s=1}^{K_0} p(s) \times \sum_{\mathcal{L} = \{l_1, \cdots, l_s\} \subseteq \mathcal{K}} \sum_{\mathbf{i} \in \mathcal{I}_{\mathcal{K}}} \int_{A_{\mathbf{i}}} \left(x - \frac{1}{s} \sum_{k=1}^{s} a_{i_{l_k}}^{(l_k)} \right)^2 f_X(x) dx. \tag{20}$$

Notice that each of the above integrals can be rewritten as follows

$$\int_{A_{\mathbf{i}}} \left(x - \frac{1}{s} \sum_{k=1}^{s} a_{i_{l_k}}^{(l_k)} \right)^2 f_X(x) dx$$
$$= \int_{A_{\mathbf{i}}} x^2 f_X(x) dx - 2 \frac{1}{s} \sum_{k=1}^{s} a_{i_{l_k}}^{(l_k)} c_{\mathbf{i}} + \left(\frac{1}{s} \sum_{k=1}^{s} a_{i_{l_k}}^{(l_k)} \right)^2 P_{\mathbf{i}}.$$

By substituting the above equation in (20) we obtain that

$$\bar{D}_1(\mathbf{f}, \mathbf{g}) = \sum_{s=1}^{K_0} p(s) \times \sum_{\mathcal{L} = \{l_1, \cdots, l_s\} \subseteq \mathcal{K}} \left(\sum_{\mathbf{i} \in \mathcal{I}_{\mathcal{K}}} \int_{A_{\mathbf{i}}} x^2 f_X(x) dx \right.$$
$$\left. -2 \sum_{\mathbf{i} \in \mathcal{I}_{\mathcal{K}}} \frac{1}{s} \sum_{k=1}^{s} a_{i_{l_k}}^{(l_k)} c_{\mathbf{i}} + \sum_{\mathbf{i} \in \mathcal{I}_{\mathcal{K}}} \left(\frac{1}{s} \sum_{k=1}^{s} a_{i_{l_k}}^{(l_k)} \right)^2 P_{\mathbf{i}} \right).$$

Further, $\bar{D}_1(\mathbf{f},\mathbf{g})$ can be rewritten as follows

$$\begin{aligned}\bar{D}_1(\mathbf{f},\mathbf{g}) &= \sum_{s=1}^{K_0} p(s)\times\binom{K}{s}\mathcal{U} - 2\sum_{s=1}^{K_0} p(s) \sum_{\mathcal{L}=\{l_1,\cdots,l_s\}\subseteq\mathcal{K}} \sum_{\mathbf{i}\in\mathcal{I}_\mathcal{K}} \frac{1}{s}\sum_{k=1}^{s} a_{i_{l_k}}^{(l_k)} c_{\mathbf{i}} \\ &+ \sum_{s=1}^{K_0} p(s) \sum_{\mathcal{L}=\{l_1,\cdots,l_s\}\subseteq\mathcal{K}} \sum_{\mathbf{i}\in\mathcal{I}_\mathcal{K}} \left(\frac{1}{s}\sum_{k=1}^{s} a_{i_{l_k}}^{(l_k)}\right)^2 P_{\mathbf{i}} \\ &= \sum_{s=1}^{K_0} p(s)\times\binom{K}{s}\mathcal{U} - 2\sum_{s=1}^{K_0} p(s) \sum_{\mathbf{i}\in\mathcal{I}_\mathcal{K}} c_{\mathbf{i}} \sum_{\mathcal{L}=\{l_1,\cdots,l_s\}\subseteq\mathcal{K}} \frac{1}{s}\sum_{k=1}^{s} a_{i_{l_k}}^{(l_k)} \\ &+ \sum_{s=1}^{K_0} p(s) \sum_{\mathbf{i}\in\mathcal{I}_\mathcal{K}} P_{\mathbf{i}} \sum_{\mathcal{L}=\{l_1,\cdots,l_s\}\subseteq\mathcal{K}} \left(\frac{1}{s}\sum_{k=1}^{s} a_{i_{l_k}}^{(l_k)}\right)^2 .\end{aligned} \tag{21}$$

We have used the fact that the number of subsets of s descriptions equals $\binom{K}{s}$. Now note that the values $\mathcal{U}$, $c_{\mathbf{i}}$ and $P_{\mathbf{i}}$ depend only on the encoder, hence they are fixed for the optimization problem. Therefore, from (21) it is already clear that $\bar{D}_1(\mathbf{f},\mathbf{g})$ is a quadratic function of $\mathbf{y}$. It follows that there exist some scalar r, some vector $\mathbf{u}$ and some matrix $\mathbf{B}$ such that (13) to hold. More specifically, there exist some $(\sum_{k=1}^K M_k)$-dimensional column vector $\mathbf{u}$ and some $(\sum_{k=1}^K M_k)\times(\sum_{k=1}^K M_k)$ symmetric matrix $\mathbf{B}$ satisfying the following equalities

$$\mathbf{u}^T\mathbf{y} = -2\sum_{s=1}^{K_0} p(s) \sum_{\mathbf{i}\in\mathcal{I}_\mathcal{K}} c_{\mathbf{i}} \underbrace{\sum_{\mathcal{L}=\{l_1,\cdots,l_s\}\subseteq\mathcal{K}} \frac{1}{s}\sum_{k=1}^{s} a_{i_{l_k}}^{(l_k)}}_{T_1(s,\mathbf{i})}, \tag{22}$$

$$\mathbf{y}^T\mathbf{B}\mathbf{y} = \sum_{s=1}^{K_0} p(s) \sum_{\mathbf{i}\in\mathcal{I}_\mathcal{K}} P_{\mathbf{i}} \underbrace{\sum_{\mathcal{L}=\{l_1,\cdots,l_s\}\subseteq\mathcal{K}} \left(\frac{1}{s}\sum_{k=1}^{s} a_{i_{l_k}}^{(l_k)}\right)^2}_{T_2(s,\mathbf{i})} . \tag{23}$$

Further, by choosing r as in (17), relation (13) follows. The proof of the fact that equalities (22) and (23) are satisfied for $\mathbf{u}$ and $\mathbf{B}$ defined in the statement of Proposition 1 is deferred to the appendix. Next we will show that matrix $\mathbf{B}$ is positive definite. Notice that matrix $\mathbf{B}$ is said to be positive definite if and only if $\mathbf{y}^T\mathbf{B}\mathbf{y} > 0$ for any $(\sum_{k=1}^K M_k)$-dimensional non-zero vector $\mathbf{y}$ [38]. It is clear from (23) that condition $\mathbf{y}^T\mathbf{B}\mathbf{y} \geq 0$ is satisfied since $p(s)$, $P_{\mathbf{i}}$ and $\left(\frac{1}{s}\sum_{k=1}^{s} a_{i_{l_k}}^{(l_k)}\right)^2$ are all non-negative values. Assume now that $\mathbf{y}^T\mathbf{B}\mathbf{y} = 0$. We assume that there is some $s_0, 1 \leq s_0 \leq K_0 - 1$, such that $p(s_0) \neq 0$. Further, since $p(s_0) \neq 0$ and $P_{\mathbf{i}} \neq 0$, (23) implies that

$$\sum_{\mathcal{L}=\{l_1,\cdots,l_{s_0}\}\subseteq\mathcal{K}} \left(\frac{1}{s_0}\sum_{k=1}^{s_0} a_{i_{l_k}}^{(l_k)}\right)^2 = 0,$$

for all $\mathbf{i} \in \mathcal{I}_\mathcal{K}$. The above equality further implies that

$$\sum_{k=1}^{s_0} a_{i_{l_k}}^{(l_k)} = 0, \tag{24}$$

for any $\mathbf{i} \in \mathcal{I}_\mathcal{K}$ and $\mathcal{L} = \{l_1,\cdots,l_{s_0}\} \subseteq \mathcal{K}$.

Consider now an arbitrary $\mathbf{i} \in \mathcal{I}_\mathcal{K}$ and two arbitrary distinct descriptions $v \neq t$, $1 \leq v,t \leq K$. Pick a description set $\mathcal{L}_1$ such that it contains the v-th description but not the t-th description. Also pick a description set $\mathcal{L}_2$ such that $\mathcal{L}_2 = (\mathcal{L}_1\backslash\{v\}) \cup \{t\}$. Subtracting equation (24) corresponding to $\mathcal{L}_1$ from equation (24) corresponding to $\mathcal{L}_2$, we obtain that $a_{i_v}^{(v)} = a_{i_t}^{(t)}$. Since the above relation holds for any v and t, using further (24) it follows that $a_{i_1}^{(1)} = a_{i_2}^{(2)} = \cdots = a_{i_K}^{(K)} = 0$. Further, since the above equalities are valid for all $\mathbf{i} \in \mathcal{I}_\mathcal{K}$, it follows that $\mathbf{y} = 0$. ■

VI. Encoder Optimization Step

At this step the decoder $\mathbf{g}$ and the vector $\mathbf{Q}$ are fixed and the encoder $\mathbf{f}$ is optimized. Here it is useful to rewrite the cost function as follows

$$\mathcal{C}(\mathbf{f},\mathbf{g},\boldsymbol{\mu},\mathbf{Q}) = \sum_{\mathbf{i}\in\mathcal{I}_\mathcal{K}} \int_{A_{\mathbf{i}}} \left(\sum_{\mathcal{L}\subseteq\mathcal{K}} p(|\mathcal{L}|) \left(x - a_{i_{l_1},\cdots,i_{l_s}}^{(\mathcal{L})}\right)^2 + \sum_{k=1}^K \log_2 \frac{1}{Q_{i_k}^{(k)}} \right) f_X(x)dx, \tag{25}$$

where $\mathcal{L} = \{l_1,\cdots,l_s\}$. The purpose of this step is to determine the sets $A_{\mathbf{i}}$ for all $\mathbf{i} \in \mathcal{I}_\mathcal{K}$, such that they form a partition of $\mathbb{R}$ and (25) is minimized. A sufficient condition for minimizing (25) is $A_{\mathbf{i}} \subseteq \mathcal{A}_{\mathbf{i}}$, where

$$\begin{aligned}\mathcal{A}_{\mathbf{i}} \triangleq \{x\in\mathbb{R}| \sum_{\mathcal{L}\subseteq\mathcal{K}} p(|\mathcal{L}|) \left(x - a_{i_{l_1},\cdots,i_{l_s}}^{(\mathcal{L})}\right)^2 + \sum_{k=1}^K \log_2 \frac{1}{Q_{i_k}^{(k)}} \leq \\ \sum_{\mathcal{L}\subseteq\mathcal{K}} p(|\mathcal{L}|) \left(x - a_{i'_{l_1},\cdots,i'_{l_s}}^{(\mathcal{L})}\right)^2 + \sum_{k=1}^K \log_2 \frac{1}{Q_{i'_k}^{(k)}}, \quad \text{for all} \quad \mathbf{i}' \in \mathcal{I}_\mathcal{K} - \{\mathbf{i}\}\}.\end{aligned} \tag{26}$$

Thus, the problem reduces to determining the sets $\mathcal{A}_{\mathbf{i}}$. Further, by denoting

$$\alpha_{\mathbf{i}} = \sum_{\mathcal{L}\subseteq\mathcal{K}} p(|\mathcal{L}|) a_{i_{l_1},\cdots,i_{l_s}}^{(\mathcal{L})} \tag{27}$$

$$\beta_{\mathbf{i}} = \sum_{\mathcal{L}\subseteq\mathcal{K}} p(|\mathcal{L}|) (a_{i_{l_1},\cdots,i_{l_s}}^{(\mathcal{L})})^2 + \sum_{k=1}^K \log_2 \frac{1}{Q_{i_k}^{(k)}}, \tag{28}$$

we can write (26) equivalently as follows

$$\begin{aligned}\mathcal{A}_{\mathbf{i}} = \{x\in\mathbb{R}| 2\alpha_{\mathbf{i}}x - \beta_{\mathbf{i}} \geq 2\alpha_{\mathbf{i}'}x - \beta_{\mathbf{i}'}, \\ \text{for all} \quad \mathbf{i}' \in \mathcal{I}_\mathcal{K} - \{\mathbf{i}\}\}.\end{aligned} \tag{29}$$

It is clear now that each set $\mathcal{A}_{\mathbf{i}}$ is either a non-empty interval on the real line or the empty-set. This justifies the claim that there is no loss in the performance of an MDSQ by letting all cells in the central partition be intervals.

The straightforward algorithm to solve (29) takes $O(N^2)$ time [5], given that the all values $\alpha_{\mathbf{i}}$ and $\beta_{\mathbf{i}}$ are already computed. On the other hand, a faster algorithm, running in $O(N)$ time, was proposed by Dumitrescu in [39]. Note that the number of terms in the summation in (27) is $2^K - 1$, while in (28) it is $K + 2^K - 1$. Additionally, the computation of each $a_{i_{l_1},\cdots,i_{l_s}}^{(\mathcal{L})}$, for $2 \leq s \leq K_0$, using (6) takes s operations. Therefore, in order to compute $\alpha_{\mathbf{i}}$ and $\beta_{\mathbf{i}}$ for fixed $\mathbf{i}$, $O(K_0 2^K)$ operations are needed. This leads to $O(NK_0 2^K)$ operations over all tuples $\mathbf{i} \in \mathcal{I}_\mathcal{K}$. We will show next that it is possible to drastically reduce this amount by exploiting the linear decoders. To this end we will derive simpler expressions for $\alpha_{\mathbf{i}}$ and $\beta_{\mathbf{i}}$. First rewrite $\alpha_{\mathbf{i}}$ as

$$\alpha_{\mathbf{i}} = \underbrace{\sum_{s=1}^{K_0} p(s) \sum_{\mathcal{L}=\{l_1,\cdots,l_s\}\subseteq\mathcal{K}} a^{(\mathcal{L})}_{i_{l_1},\cdots,i_{l_s}}}_{T_3(\mathbf{i})} + \sum_{s=K_0+1}^{K} \sum_{\mathcal{L}=\{l_1,\cdots,l_s\}\subseteq\mathcal{K}} p(s) a^{(\mathcal{L})}_{i_{l_1},\cdots,i_{l_s}}. \tag{30}$$

Further, $T_3(\mathbf{i})$ can be written as follows

$$T_3(\mathbf{i}) = \sum_{s=1}^{K_0} p(s) T_1(s,\mathbf{i}) = \sum_{s=1}^{K_0} p(s) \frac{1}{K}\binom{K}{s} \sum_{j=1}^{K} a^{(j)}_{i_j},$$

where $T_1(s,\mathbf{i})$ was defined in equation (22) and simplified in the appendix in equation (36). Substituting further in (30) we obtain

$$\alpha_{\mathbf{i}} = \sum_{s=1}^{K_0} p(s) \frac{1}{K}\binom{K}{s} \sum_{j=1}^{K} a^{(j)}_{i_j} + \sum_{s=K_0+1}^{K} \sum_{\mathcal{L}=\{l_1,\cdots,l_s\}\subseteq\mathcal{K}} p(s) a^{(\mathcal{L})}_{i_{l_1},\cdots,i_{l_s}}. \tag{31}$$

Since the quantity $\sum_{s=1}^{K_0} p(s)\frac{1}{K}\binom{K}{s}$ can be precomputed, rather than evaluated for each $\mathbf{i}$, the number of operations in the first term equals $K-1$ additions plus 1 multiplication. Hence K operations are needed for the first term. The number of operations in the second term is $2\sum_{s=K_0+1}^{K}\binom{K}{s} - 1$, since there are $\sum_{s=K_0+1}^{K}\binom{K}{s}$ terms in the summation. Therefore, $K + 2\sum_{s=K_0+1}^{K}\binom{K}{s} - 1$ operations are needed to compute each $\alpha_{\mathbf{i}}$. When $K_0 = K$, the number of operations is K. When $K_0 = K-1$, the number of operations is $K+1$.

Notice that the second summation in (28) can be computed in $O(K)$ time. Let us denote by $\beta'_{\mathbf{i}}$ the first summation in (28). To simplify the expression of $\beta'_{\mathbf{i}}$ note first that

$$\begin{aligned}\beta'_{\mathbf{i}} &= \sum_{s=1}^{K} p(s) \sum_{\mathcal{L}=\{l_1,\cdots,l_s\}\subseteq\mathcal{K}} (a^{(\mathcal{L})}_{i_{l_1},\cdots,i_{l_s}})^2 \\ &= \underbrace{\sum_{s=1}^{K_0} p(s) \sum_{\mathcal{L}=\{l_1,\cdots,l_s\}\subseteq\mathcal{K}} (a^{(\mathcal{L})}_{i_{l_1},\cdots,i_{l_s}})^2}_{T_4(\mathbf{i})} \\ &\quad + \sum_{s=K_0+1}^{K} \sum_{\mathcal{L}=\{l_1,\cdots,l_s\}\subseteq\mathcal{K}} p(s) (a^{(\mathcal{L})}_{i_{l_1},\cdots,i_{l_s}})^2.\end{aligned} \tag{32}$$

Further, note that $T_4(\mathbf{i})$ can be written as

$$\begin{aligned}T_4(\mathbf{i}) &= \sum_{s=1}^{K_0} p(s) T_2(s,\mathbf{i}) \\ &= \sum_{s=1}^{K_0} p(s) \Big(\frac{1}{Ks}\binom{K}{s} \sum_{j=1}^{K} (a^{(j)}_{i_j})^2 \\ &\quad + \frac{2(s-1)}{Ks(K-1)}\binom{K}{s} \sum_{j=1}^{K-1}\sum_{j'=j+1}^{K} a^{(j)}_{i_j} a^{(j')}_{i_{j'}} \Big) \\ &= \sum_{s=1}^{K_0} p(s) \frac{1}{Ks}\binom{K}{s} \sum_{j=1}^{K} (a^{(j)}_{i_j})^2 \\ &\quad + \sum_{s=1}^{K_0} p(s) \frac{2(s-1)}{Ks(K-1)}\binom{K}{s} \sum_{j=1}^{K-1}\sum_{j'=j+1}^{K} a^{(j)}_{i_j} a^{(j')}_{i_{j'}},\end{aligned}$$

where $T_2(s,\mathbf{i})$ was defined in equation (23). Additionally, equation (38) in the appendix was used to establish the second equality in the above sequence. Substituting further in (32) we obtain

$$\begin{aligned}\beta'_{\mathbf{i}} &= \sum_{s=1}^{K_0} p(s) \frac{1}{Ks}\binom{K}{s} \sum_{j=1}^{K} (a^{(j)}_{i_j})^2 \\ &\quad + \sum_{s=1}^{K_0} p(s) \frac{2(s-1)}{Ks(K-1)}\binom{K}{s} \sum_{j=1}^{K-1}\sum_{j'=j+1}^{K} a^{(j)}_{i_j} a^{(j')}_{i_{j'}} \\ &\quad + \sum_{s=K_0+1}^{K} \sum_{\mathcal{L}=\{l_1,\cdots,l_s\}\subseteq\mathcal{K}} p(s) (a^{(\mathcal{L})}_{i_{l_1},\cdots,i_{l_s}})^2.\end{aligned}$$

Since the quantities $\sum_{s=1}^{K_0} p(s)\frac{1}{Ks}\binom{K}{s}$ and $\sum_{s=1}^{K_0} p(s)\frac{2(s-1)}{Ks(K-1)}\binom{K}{s}$ can be precomputed, rather than evaluated for each $\mathbf{i}$, the number of operations in the first term equals $K-1$ additions plus $K+1$ multiplications, hence $2K$ operations are needed. The number of operations in the second term equals $\frac{1}{2}(K^2-K)-1$ additions plus $\frac{1}{2}(K^2-K)+1$ multiplications, hence (K^2-K) operations are needed. The number of operations in the third term is 0 when $K_0 = K$, and it is $3\sum_{s=K_0+1}^{K}\binom{K}{s} - 1$, when $K_0 < K$, since there are $\sum_{s=K_0+1}^{K}\binom{K}{s}$ terms in the summation. Let $k_0 \triangleq K - K_0$ and assume that k_0 is a small positive constant (thus $k_0 << K$). Then

$$\sum_{s=K_0+1}^{K}\binom{K}{s} = \sum_{t=0}^{k_0-1}\binom{K}{t} \le k_0\binom{K}{k_0-1} = O(K^{k_0-1}).$$

Therefore, the number of operations needed to compute each $\beta_{\mathbf{i}}$ is $O(K^2)$ when $k_0 \le 3$ and it is $O(K^{k_0-1})$ for $k_0 > 3$. Additionally, based on the discussion below equation (31), we obtain that the number of operations to compute each $\alpha_{\mathbf{i}}$ is $O(K^{\max(1,k_0-1)})$. In conclusion, the number of operations to obtain the $\alpha_{\mathbf{i}}$'s and $\beta_{\mathbf{i}}$'s, for all $\mathbf{i} \in \mathcal{I}_{\mathcal{K}}$ is $O(NK^{\max(2,k_0-1)})$. This represents a significant reduction from $O(NK_0 2^K)$.

VII. Experimental Results

The purpose of this section is to assess the practical performance of the proposed EC-LMDSQ scheme in comparison with MDLVQ and UEP. We consider a zero mean, unit variance, memoryless Gaussian source truncated to the interval $[-6.0, 6.0]$. We will compare the performance of the aforementioned schemes for $K = 4$. In all cases we assume transmission over independent channels with the same probability q of failure. Consequently, the probability $p(s)$ of receiving a particular set of s descriptions is $p(s) = (1-q)^s q^{K-s}$, for $s, 0 \le s \le K$. We will consider three values for q, namely $q = 0.01, 0.05, 0.075$. For each q we will plot the performance of each MD technique, measured using the expected distortion in dB, i.e., $10\log_{10}\bar{D}$, versus the average rate R of the side descriptions.

To assess the performance of MDLVQ we will use the expressions of the distortions at the side and joint decoders derived in [24] under the assumption of high rate and high reuse index. Recall that the reuse index represents how many times each index of a description is assigned a central lattice point. The reuse index I controls the trade-off between the central distortion and the distortions at the remaining decoders. Therefore, it has to be optimized in order to achieve the smallest expected distortion for given q and rate R. Based on the asymptotical expression of the distortions derived in [24] the authors of [30] have computed the optimal value I_{opt} and the corresponding optimal expected distortion for an MDLVQ of dimension n [30, Eq. (24)][3], denoted by $\bar{D}_{latt}(n)$, as follows

$$\bar{D}_{latt}(n) = q^K + 2\pi e(1-q^K)\Big(G(\Lambda_c)2^{-2KR} + qy(q)\frac{G(S_n)}{K^2}2^{-2R} + qz(q)G(S_{Kn-n})K^{-\frac{K}{K-1}}\Big), \quad \text{if } R < R_0(K,q), \tag{33}$$

$$\bar{D}_{latt}(n) = q^K + 2\pi e(1-q^K)q^{\frac{K-1}{K}}2^{-2R}\left(q^{\frac{1}{K}}y(q)\frac{G(S_n)}{K^2} + G(\Lambda_c)^{\frac{1}{K}}\left(\frac{z(q)G(S_{Kn-n})}{K-1}\right)^{\frac{K-1}{K}}\right) \text{ if } R \geq R_0(K,q), \tag{34}$$

where $y(q) \triangleq \frac{1}{1-q^K}\sum_{k=1}^{K-1}\binom{K}{k}(1-q)^k q^{K-1-k}$, $z(q) \triangleq \frac{1}{1-q^K}\sum_{k=1}^{K-1}\binom{K}{k}(1-q)^k q^{K-1-k}\frac{K-k}{k}$, and

$$R_0(K,q) \triangleq \frac{1}{2K}\log_2\frac{G(\Lambda_c)K^{\frac{K}{K-1}}(K-1)}{qz(q)G(S_{Kn-n})}. \tag{35}$$

Moreover, $G(\Lambda_c)$, respectively $G(S_t)$, denotes the normalized second moment of the central lattice Λ_c, respectively of a sphere in $\mathbb{R}^t$ for $t \geq 1$.

In our comparison we will use MDLVQ with $n = 1$ and $n \to \infty$. The performance when $n \to \infty$ is measured using $\bar{D}_{latt}(\infty) \triangleq \lim_{n\to\infty}\bar{D}_{latt}(n)$. It is known that there is a sequence of lattices $\Lambda_n \subset \mathbb{R}^n$ such that $\lim_{n\to\infty}G(\Lambda_n) = \frac{1}{2\pi e}$ [42]. Additionally, we also have $\lim_{n\to\infty}G(S_n) = \frac{1}{2\pi e}$. Then $\bar{D}_{latt}(\infty)$ can be obtained by replacing $G(\Lambda_c)$, $G(S_n)$ and $G(S_{Kn-n})$ by $\frac{1}{2\pi e}$ in (33), (34) and (35). In our analysis we will use the acronym MDLVQ-A1, respectively MDLVQ-Ainf to refer to the MDLVQ with $n = 1$, respectively $n \to \infty$, with the asymptotical assessment of the performance as explained above.

Since the expression of the expected distortion of an MDLVQ given by (33) and (34) is based on the assumption that both R and I approach ∞, it is not accurate at low rates and low values of I. Therefore, for fairness of comparison we will also compute the actual expected distortion of an MDLVQ of dimension $n = 1$ for the Gaussian source truncated to the interval $[-6, 6]$. We will use the scheme proposed in [25] for MDLVQ of dimension 1, where the side codebooks are shifts of the side lattice. The authors of [25] argue that using different shifts of the side lattice as side codebooks, as opposed to having all side codebooks coinciding with the side lattice, improves the performance of the MDLVQ and demonstrate their claim empirically. The authors of [25] also propose IA's which are optimal when R and I approach infinity. We point out that the IA's designed in [25] only allow values of I which are multiples of 4. We will consider four values for I, namely $4, 8, 12$ and 16. For each I we consider all values of N which are multiples of $I/4$ up to $N = 48 \times I/4$. For each N, the size of the interval in the central partition will be $12/N$. The index assignment for each pair (N, I) will determine the number of cells in each side quantizer. After computing the expected distortion and the average rate for each MDLVQ generated in this way, the points on the lower boundary of the set of pairs consisting of the expected distortion and the average rate, will be selected and used to illustrate the performance. Additionally, we emphasize that the average rate is computed as the average of the entropies of the side descriptions. We will use the acronym MDLVQ-1 to refer to the MDLVQ of dimension 1 with the performance measured as described above.

To assess the performance of the proposed EC-LMDSQ scheme we will run the algorithm for minimizing the cost function $\mathcal{C}(\mathbf{f}, \mathbf{g}, \boldsymbol{\mu}, \mathbf{Q})$ with $\mu_1 = \mu_2 = \mu_3 = \mu_4$, for various values of μ_1 in the range $[0, 0.1]$. The number of central cells N and the index assignment used at initialization are obtained as described above, but for a smaller set of values of N. In particular, we consider $N = k \times I/4$, where $k = 3, 7, 11, 15, 19, 23, 27, 31, 35, 39, 43, 47, 51, 59, 67, 75, 83$ when $K_0 = 3$, and $k = 3, 11, 19, 27, 35, 43, 51, 59, 67, 75, 83$ when $K_0 = 2$. The index assignment for each pair (N, I) will determine the number of cells in each side quantizer. The algorithm is run for each initial configuration and is stopped when the relative difference in the cost function $\mathcal{C}(\mathbf{f}, \mathbf{g}, \boldsymbol{\mu}, \mathbf{Q})$ becomes smaller than 5×10^{-5}. For each obtained EC-LMDSQ the expected distortion and the average rate are computed. Then the lower boundary of the set of obtained rate-distortion points is used to illustrate the performance.

Finally, to asses the performance of the UEP scheme we consider optimized erasure protection such that the expected distortion to be minimized given the constraint on the rate. The expected distortion is expressed in terms of the operational rate-distortion (RD) function of the SRC coder, denoted by $D_o(R)$. To be more specific, $D_o(R)$ represents the distortion achieved if only a prefix of rate R is decoded. The SRC can be regarded as a sequence of embedded vector quantizers of dimension n. We will consider the cases of $n = 1$ and $n \to \infty$, referred to as UEP-1, respectively UEP-inf. In the latter case, since the Gaussian source is known to be successively refinable, the value of the operational RD function equals exactly the information theoretical RD function, i.e., we have $D_{o,n\to\infty}(R) = 2^{-2R}$. For UEP-1 we assume that the SRC is formed of a sequence of entropy-constrained embedded uniform quantizers, and use the high rate approximation of the distortion, leading to $D_{o,n=1}(R) = \frac{2\pi e}{12}2^{-2R}$. Finally, the

[3]We point out that there is a typo in [30, Eq. (24)], in the expression of $D_4(L, R, p)$ for $R < R_4(L, P)$. Namely, the exponent $-\frac{L-1}{L}$ has to be replaced by $-\frac{L}{L-1}$. Additionally, note that the quantity $D_4(L, R, p)$ mentioned above represents the expected distortion conditioned on the event that at least one description is received.

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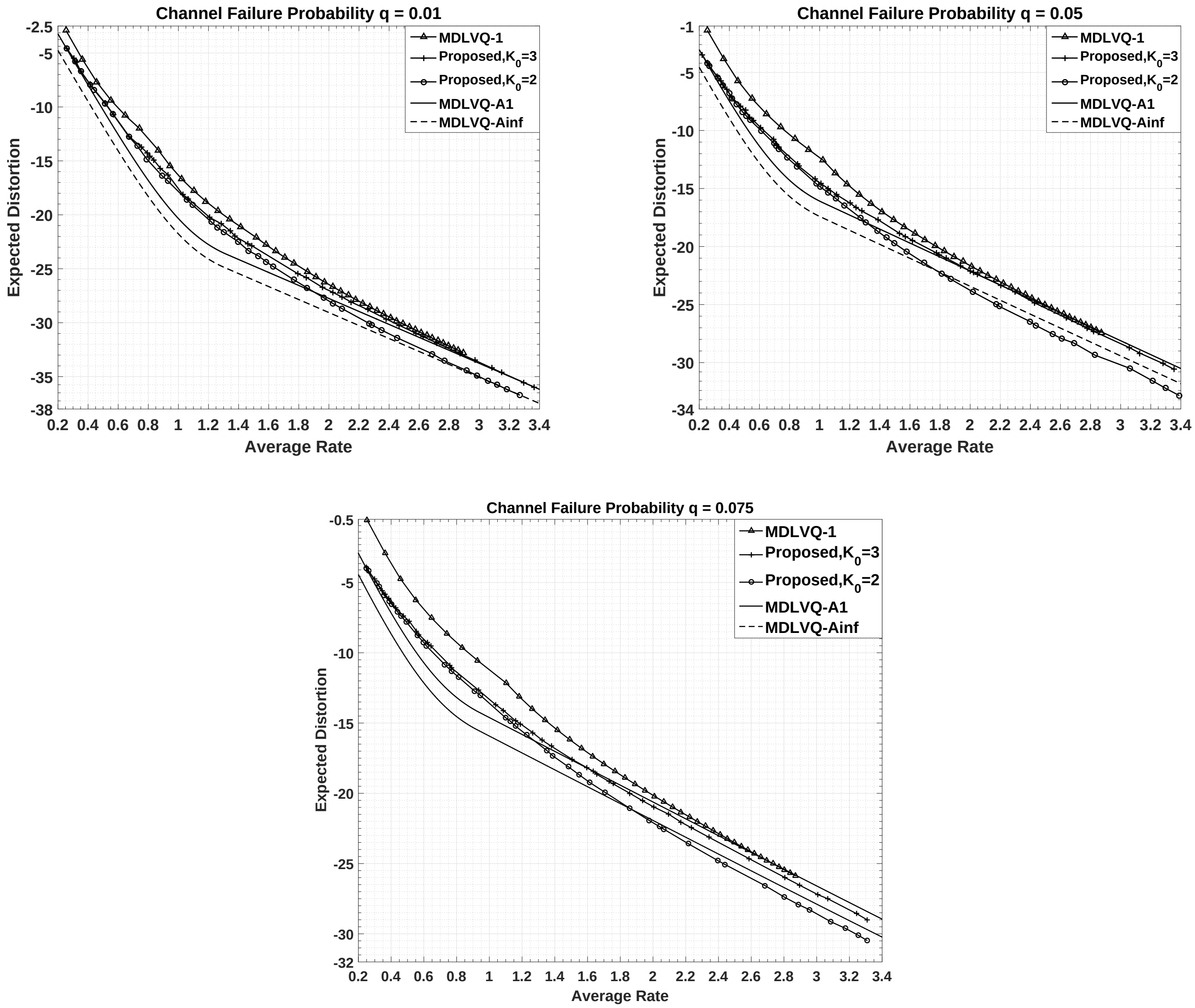

Fig. 1: Performance comparison with MDLVQ for $q = 0.01$ (top left), $q = 0.05$ (top right) and $q = 0.075$ (bottom).

associated optimization problem and its solution are discussed in [30]. We use the results in [30] to plot the performance for UEP-1 and UEP-inf.

Figure 1 illustrates the performance of EC-LMDSQ in comparison with MDLVQ-1, MDLVQ-A1 and MDLVQ-Ainf, for $q = 0.01, 0.05, 0.075$. The interval of rates covered is approximately from 0.2 to 3.4. We note that EC-LMDSQ with $K_0 = 3$ outperforms MDLVQ-1 for the whole range of rates and the gap tends to increase as the rate decreases. Additionally, we observe that given a fixed rate value, the gap is generally higher for larger q. In particular, for rates up to 1, the gap is at least 0.8 dB, 1.5 dB, respectively 2 dB, for $q = 0.01, 0.05, 0.075$, respectively. On the other hand, as the rate increases, the gap decreases and for the smaller values of q (i.e., 0.01 and 0.05) it eventually becomes negligible.

Notice that MDLVQ-A1 is better than MDLVQ-1, but their performance becomes close as the rate approaches 2. We believe that this is an indication of the fact that the approximations made under the high rate assumption when computing the distortions for MDLVQ-A1 are not accurate enough for rates smaller than 2, but become accurate for rates higher than 2.

We see that EC-LMDSQ with $K_0 = 3$ has performance very close or better than MDLVQ-A1 for rates higher than some value, which depends on q. The two schemes also have similar performance when the rates are very small (smaller than about 0.4). MDLVQ-Ainf is better than MDLVQ-A1, as expected, the gain being caused by the space-filling advantage as the dimension of the quantizer increases. Thus, MDLVQ-Ainf also beats EC-LMDSQ with $K_0 = 3$.

The performance of EC-LMDSQ with $K_0 = 2$ and with $K_0 = 3$ are very close for rates up to some value, after which the scheme with $K_0 = 2$ becomes superior. Furthermore, as the rate increases further, EC-LMDSQ with $K_0 = 2$ beats MDLVQ-A1 and eventually matches and even outperforms MDLVQ-Ainf. The latter result indicates that the gain achieved by adding optimal codebooks when three descriptions are received is higher than the gain obtained only by increasing

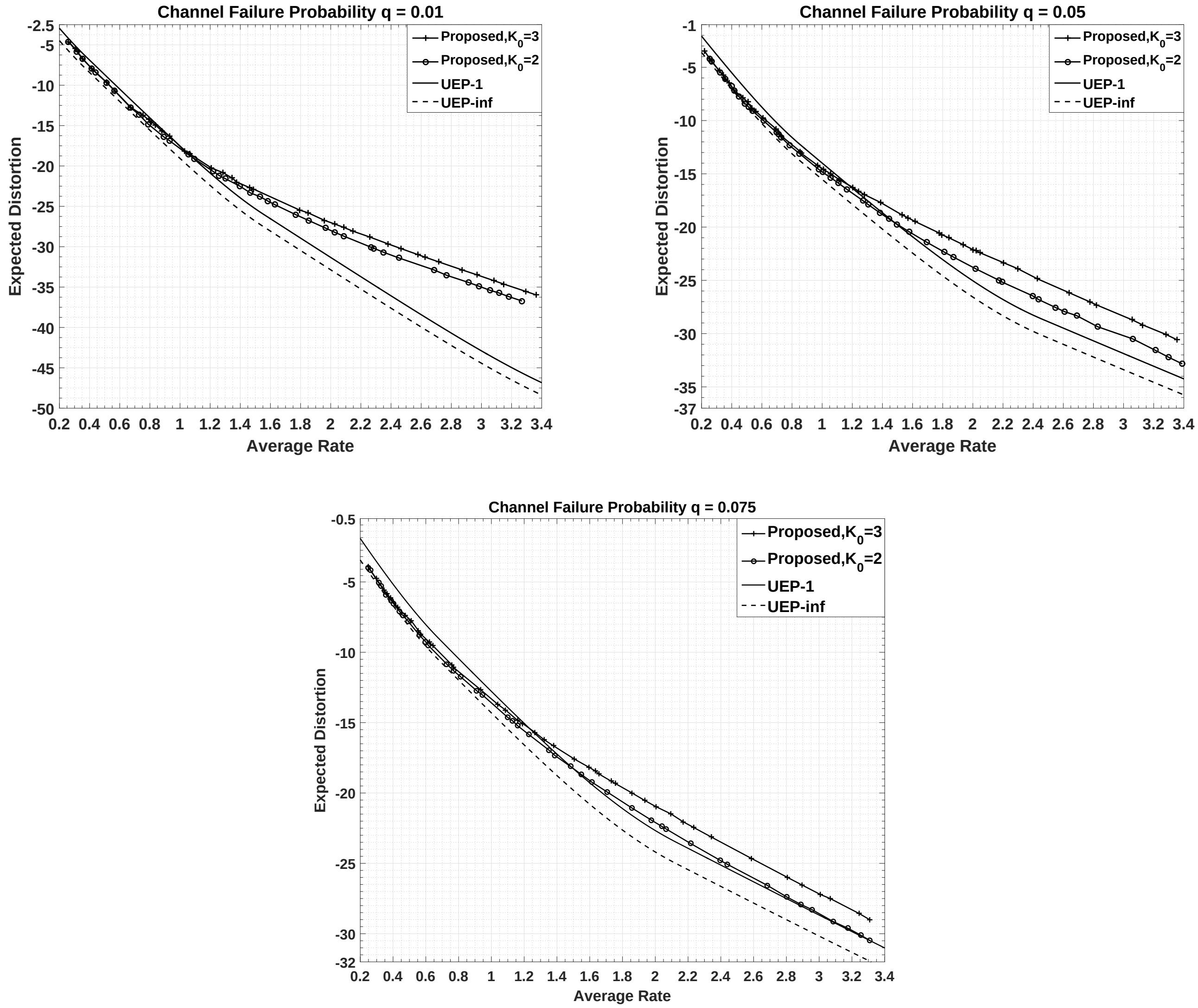

Fig. 2: Performance comparison with UEP for $q = 0.01$ (top left), $q = 0.05$ (top right) and $q = 0.075$ (bottom).

the quantization dimension without the additional codebooks.

Figure 2 illustrates the performance of EC-LMDSQ in comparison with UEP-1 and UEP-inf, for $q = 0.01, 0.05, 0.075$. We notice that EC-LMDSQ outperforms UEP-1 for rates smaller than some value r_1, which depends on q and K_0. This improvement over UEP-1 for small rates is larger for the larger values of q. Notably, for $q = 0.05$ and $q = 0.075$, and rates smaller than about 0.5, EC-LMDSQ has performance extremely close to UEP-inf. Furthermore, it is worth noting that for rates higher than r_1 the gap between EC-LMDSQ and UEP-1 increases as q decreases and this variation is quite dramatic. In particular, for $q = 0.01$ the gap between EC-LMDSQ with $K_0 = 2$ and UEP-1 increases consistently up to about 9 dB, while for $q = 0.075$ the largest gap is not higher than 0.5 dB and it actually vanishes for rates larger than 2.8. We point out that, although the theoretical optimum for MDC scheme in the symmetric case is not known (even for a Gaussian source), it was proved that UEP-inf is only a constant bit-rate away from the optimum [40]. An upper bound on this gap for $K = 4$ is 0.865 [40].

While the expected distortion of EC-LMDSQ is always smaller than that of MDLVQ-1, it is also interesting to compare the distortions when only k descriptions are received, denoted by D_k, for each $k, 1 \leq k \leq 4$. It is known that there are trade-offs between the performance of level k decoders (i.e. k-description decoders) for different values of k. In other words, if the performance of some levels improves, it may become worse at some other levels. Figure 3 plots the values of D_k for EC-LMDSQ in comparison with MDLVQ-1 for $q = 0.05$. We notice that the performance of EC-LMDSQ with $K_0 = 3$ is always better than that of MDLVQ-1 for level $1, 2$ and 3 decoders, and the difference increases as the rate decreases reaching values of about $9.4, 7.7$ and 5.3 dB for levels $1, 2, 3$ respectively. On the other hand, the central distortion D_4 of MDLVQ-1 is slightly smaller than that of the proposed schemes when the rate is sufficiently high. Furthermore, it can be seen that the value of D_3 for EC-LMDSQ with $K_0 = 2$, decreases significantly in comparison

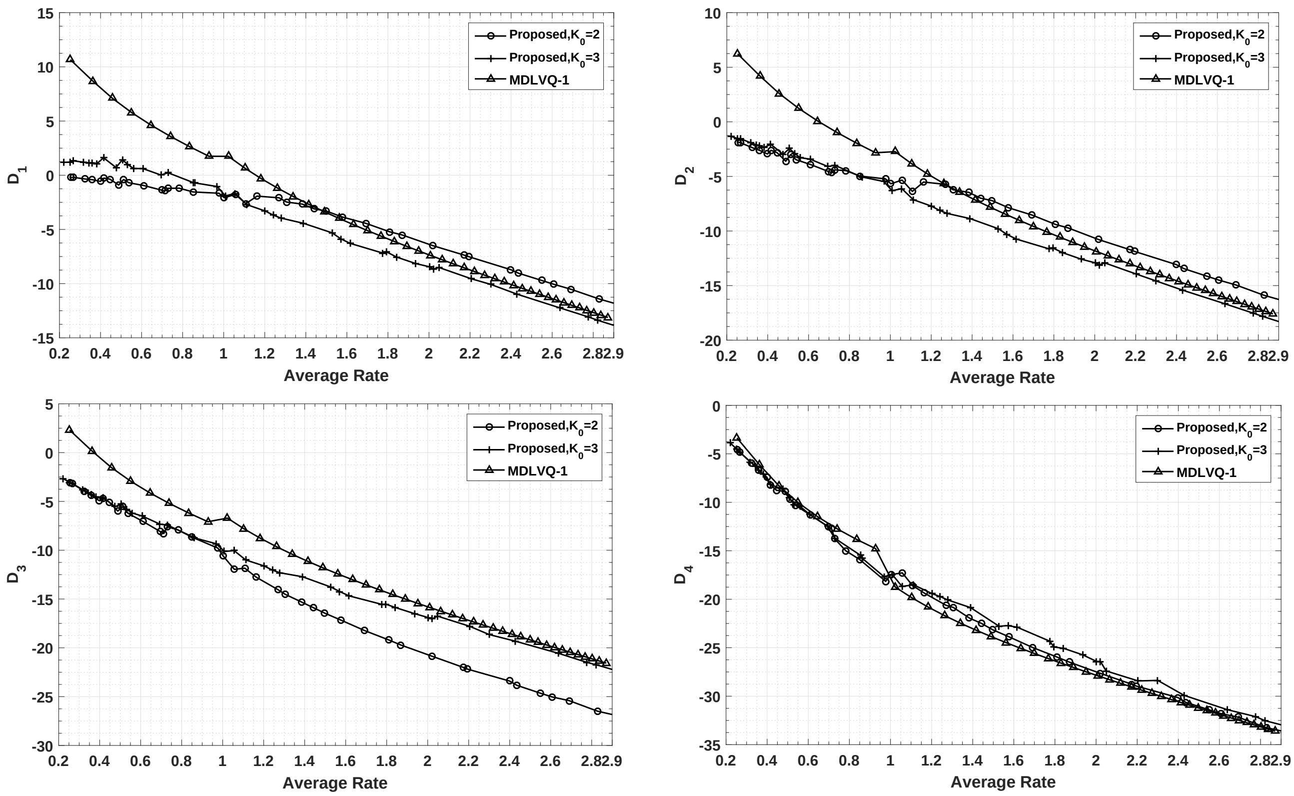

Fig. 3: Average distortions when k descriptions are received, $1 \leq k \leq K$, and $q = 0.05$.

with $K_0 = 3$, for rates higher than 1. This is the effect of using optimal reconstruction values for level 3 decoders in the former case. On the other hand, as a result of the trade-off between the performance at different levels, the values of D_1 and D_2 for $K_0 = 2$ becomes larger than for $K_0 = 3$ for rates higher than 1, and, as the rate increases further, even higher than for MDLVQ-1. However, the overall performance in terms of expected distortion is better for the former scheme than for the latter two, as seen in Figure 1.

VIII. Conclusion

In a traditional multiple description scalar quantizer (MDSQ) with optimized codebooks all codebooks have to be stored at the decoder. Thus, the memory requirement at the decoder increases exponentially with the number of descriptions. This poses a problem in applications where the memory resources are scarce, such as transmission to mobile devices. To alleviate this problem we propose the use of entropy-constrained MDSQ with linear joint decoders (EC-LMDSQ). The decoder of an EC-LMDSQ stores all side codebooks and a few joint codebooks, while generating the other joint codebooks using averages of the side reconstruction values. This way, the storage space may be significantly reduced.

We further propose an algorithm for symmetric EC-LMDSQ design, which locally minimizes the Lagrangian formed as a weighted sum of the expected distortion and of the side encoders' rates. The algorithm is similar in spirit to the EC-MDSQ design algorithm of Vaishampayan and Domaszewicz for two descriptions, however, because of the linear joint decoders, the decoder optimization step is more complex since the reconstruction values can no longer be optimized separately. However, we overcome this challenge by proving that the problem is convex quadratic and, therefore, easily solvable. Additionally, we show how the complexity of the encoder optimization step can be drastically reduced by exploiting the particular form of the cost function.

We assess empirically the performance of the proposed EC-LMDSQ design algorithm in comparison with state of the art MDC schemes with reduced storage space at decoder, such as multiple description lattice vector quantizers (MDLVQ) and the technique based on unequal erasure protection (UEP). Our experimental results reveal that EC-LMDSQ performs better than MDLVQ with dimension 1 quantization for all rates tested, as expected, and it also outperforms MDLVQ with infinite dimension quantization when more codebooks are added and the rate is sufficiently high. Furthermore, the proposed scheme is also superior to UEP with dimension 1 quantization at low rates.

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APPENDIX

In this appendix we complete the proof of Proposition 1 by showing that relations (22) and (23) are satisfied for $\mathbf{u}$ and $\mathbf{B}$ defined in the statement of the proposition. For this we need first to write $T_1(s, \mathbf{i})$ and $T_2(s, \mathbf{i})$ in a simpler form. Notice that by expanding $T_1(s, \mathbf{i})$, each $a_{i_j}^{(j)}$, $1 \leq j \leq K$, appears $\binom{K-1}{s-1} = \frac{s}{K}\binom{K}{s}$ times in the summation. Hence,

$$T_1(s, \mathbf{i}) = \sum_{\mathcal{L}=\{l_1,\cdots,l_s\}\subseteq\mathcal{K}} \frac{1}{s}\sum_{k=1}^{s} a_{i_{l_k}}^{(l_k)} = \frac{1}{K}\binom{K}{s}\sum_{j=1}^{K} a_{i_j}^{(j)}. \tag{36}$$

By substituting (36) in (22), relation (22) becomes equivalent to

$$\mathbf{u}^T\mathbf{y} = -2\sum_{s=1}^{K_0} p(s) \sum_{\mathbf{i}\in\mathcal{I}_\mathcal{K}} c_{\mathbf{i}} \frac{1}{K}\binom{K}{s}\sum_{j=1}^{K} a_{i_j}^{(j)}.$$

It is easy to see that the above relation is satisfied for $\mathbf{u}$ defined in (18). In order to simplify $T_2(s, \mathbf{i})$ we have to treat separately the cases $s > 1$ and $s = 1$. When $s = 1$, we have clearly

$$T_2(1, \mathbf{i}) = \sum_{j=1}^{K} (a_{i_j}^{(j)})^2. \tag{37}$$

When $s > 1$, notice that after expanding the summations, $T_2(s, \mathbf{i})$ contains terms of the form $(a_{i_j}^{(j)})^2$ and $2a_{i_j}^{(j)}a_{i_{j'}}^{(j')}$, $1 \leq j < j' \leq K$. Each $(a_{i_j}^{(j)})^2$ appears $\binom{K-1}{s-1} = \frac{s}{K}\binom{K}{s}$ times, and

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each $2a_{i_j}^{(j)}a_{i_{j'}}^{(j')}$ occurs $\binom{K-2}{s-2} = \binom{s}{2}\binom{K}{s}/\binom{K}{2}$ times. Then

$$
\begin{aligned}
T_2(s,\mathbf{i}) &= \sum_{\mathcal{L}=\{l_1,\cdots,l_s\}\subseteq\mathcal{K}} \left(\frac{1}{s}\sum_{k=1}^{s} a_{i_{l_k}}^{(l_k)}\right)^2 \\
&= \frac{1}{Ks}\binom{K}{s}\sum_{j=1}^{K}(a_{i_j}^{(j)})^2 + \frac{2\binom{s}{2}}{\binom{K}{2}s^2}\binom{K}{s}\sum_{j=1}^{K-1}\sum_{j'=j+1}^{K} a_{i_j}^{(j)}a_{i_{j'}}^{(j')} \\
&= \frac{1}{Ks}\binom{K}{s}\sum_{j=1}^{K}(a_{i_j}^{(j)})^2 + \frac{2(s-1)}{Ks(K-1)}\binom{K}{s}\sum_{j=1}^{K-1}\sum_{j'=j+1}^{K} a_{i_j}^{(j)}a_{i_{j'}}^{(j')}.
\end{aligned} \tag{38}
$$

Relation (37) implies that the above simplified expression of $T_2(s)$ also holds for $s = 1$. Substituting (38) into (23), it follows that (23) is equivalent to

$$
\mathbf{y}^T\mathbf{B}\mathbf{y} = \sum_{s=1}^{K_0} p(s) \sum_{\mathbf{i}\in\mathcal{I}_\mathcal{K}} P_{\mathbf{i}} \times
$$

$$
\left(\frac{1}{Ks}\binom{K}{s}\sum_{j=1}^{K}(a_{i_j}^{(j)})^2 + \frac{2(s-1)}{Ks(K-1)}\binom{K}{s}\sum_{j=1}^{K-1}\sum_{j'=j+1}^{K} a_{i_j}^{(j)}a_{i_{j'}}^{(j')}\right).
$$

It can be easily checked that the above equality holds for $\mathbf{B}$ defined in (19).

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