# Flexible Symmetric Multiple Description Lattice Vector Quantizer with $L \geq 3$ Descriptions

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**Abstract**

In the previous work on multiple description lattice vector quantizers (MDLVQ) with $L \geq 3$ descriptions, once the central and side lattice codebooks are fixed, the decoding quality is determined for all numbers $k$ of received descriptions. Therefore, it is not possible to achieve tradeoffs between the quality of reconstruction for different values of $k, 1 \leq k \leq L-1$. This work proposes a flexible MDLVQ capable of overcoming the above drawback. For this, a different reconstruction method is employed and a heuristic index assignment (IA) algorithm, which uses $L-2$ parameters to control the distortions for $2 \leq k \leq L-1$, is developed. Experimental results show that the proposed MDLVQ beside achieving the desired tradeoffs, significantly outperforms the classic MD scheme based on unequal erasure protection.

The second contribution of this work is a structured IA for the case $L = 3$ and the derivation of the corresponding expressions of the distortions at high resolution. The proposed IA has a simple mechanism for controlling the tradeoff between the reconstruction quality for $k = 1, 2$. The IA is able to achieve a wide range of distortion values, while keeping the product of the distortions for $k = 1, 2$ the same as in the prior work.

**Index Terms**

Multiple description coding, lattice quantization, high resolution analysis.

## I. Introduction

A multiple description (MD) coder consists of $L$ encoders for some $L \geq 2$, each encoder generating a separate description of the signal. Each description is sent to the destination over

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a separate channel, which either transmits the whole description correctly or breaks down. The decoder is able to reconstruct the source to some quality from any subset of received descriptions, while the fidelity of the reconstruction generally increases with the number of received descriptions.

An $n$-dimensional multiple description lattice vector quantizer (MDLVQ) is an MD coder consisting of a so-called central lattice $\Lambda_c \subset \mathbb{R}^n$, a so-called side lattice $\Lambda_s$, which is a sublattice of $\Lambda_c$, and an injective mapping $\alpha : \Lambda_c \to \Lambda_s^L$, which assigns to each central lattice point an $L$-tuple of side lattice points. For each $\lambda_c \in \Lambda_c$, let $\alpha_i(\lambda_c)$ denote the $i$-th component of the $L$-tuple $\alpha(\lambda_c)$, $1 \leq i \leq n$. The encoder of the MDLVQ quantizes the input vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ to the closest central lattice point $\lambda_c$ and outputs $\alpha_i(\lambda_c)$ as the $i$-th description, for $1 \leq i \leq L$. When all descriptions $\lambda_1, \cdots, \lambda_L$ are received at destination the decoder is able to uniquely identify the corresponding central lattice point and uses it as the source reconstruction. On the other hand, when only one description $\lambda_i$ is received, the decoder uses it to reconstruct the source. When all descriptions are lost the decoder outputs the source mean. Notice that when $L = 2$, only the above mentioned situations are possible at the decoder. On the other hand, when $L > 2$, it is possible that the number $k$ of descriptions received to be larger than $1$ and smaller than $L$. In such a case an appropriate decoding rule has to be specified.

The MDLVQ framework for $L = 2$ was introduced and analyzed in [1]. The authors of [1] propose a general method for the index assignment (IA) design and derive analytical expressions for the distortions at high resolution, using the squared distance as a distortion measure. While the aforementioned work addresses the symmetric case, i.e., where the rates, respectively the distortions, of the two descriptions are equal, the asymmetric case was considered in [2]. In [3], [4], the authors achieve the tradeoffs between central and side distortions by modifying the encoding rule of [1]. Additionally, an extension to more than two descriptions is briefly mentioned in [3]. A systematic study of an MDLVQ system for $L > 2$ descriptions, in the symmetric case, is performed in [5]. The authors of [5] use the setup described in the previous paragraph and for the additional situations arisen at the decoder, i.e., when the number of received descriptions is $k, 2 \leq k \leq L - 1$, use the arithmetic average of the received descriptions to reconstruct the source. They propose an IA algorithm and derive asymptotical expressions of the distortions for different numbers of received descriptions. The authors of [6] propose a simpler and faster IA algorithm and prove its optimality for $L = 2$ with any $N$, and for $L > 2$ when $N \to \infty$, where

$N$ is the index of the side lattice with respect to the central lattice. They also derive asymptotical expressions of the distortions. The work [7] proposes algorithms to improve the IA for finite $N$. The work [8] uses a simpler method to analyze the asymptotical performance of the MDLVQ of [6]. A multiple description scalar quantizer with translated lattice codebooks and the associated optimal IA are discussed in [9]. In [10] an asymmetric MDLVQ scheme with $L \geq 2$, which uses the weighted average of the received descriptions as reconstruction for $k < L$, is investigated.

It is worth pointing out that there are non IA-based MDLVQ schemes in the literature for $L = 2$ [11], [12], and for general $L$ when only the distortions of individual descriptions and of all descriptions are of interest [13]. In this work we are concerned only with IA-based MDLVQ, and will omit the specification "IA-based" in the sequel.

We emphasize that the MDLVQ framework for $L \geq 3$ considered in prior work is able to achieve tradeoffs between the reconstruction quality when all descriptions are received versus the case when only a subset of them are received, by varying the value of $N$ for fixed rate $R$ of individual description. However, it is not possible to achieve tradeoffs between the decoding quality for different values of $k$, $1 \leq k \leq L-1$.

In this work we propose a flexible MDLVQ system for $L \geq 3$, able to adjust the decoding quality for various values of $k, 1 \leq k \leq L-1$. To this aim we use a different reconstruction method and propose a heuristic IA algorithm, which uses $L-2$ parameters to control the reconstruction quality for $2 \leq k \leq L-1$. Our simulations for $L = 3, 4$ not only show that the proposed framework can achieve the desired tradeoffs, but also that it significantly outperforms the MD scheme based on successively refinable codes and unequal erasure protection, referred to as UEP [14], [15]. This result is very important since UEP is one of the few MD codes that are practical and ensure flexibility in adjusting the distortions for different numbers of received descriptions [16]. While there are theoretical constructions that outperform the UEP scheme [17]–[19], they rely on the binning technique, which is difficult to implement in practice. Additionally, to add to the merit of UEP as a term of comparison we point out that, even if it is suboptimal, UEP was shown to be only at most 0.73 bits away from the theoretically optimal description rate [20].

In the second part of this work we propose a structured IA for the case $L = 3$ with a simple mechanism to control the tradeoffs between the distortions for one, respectively two, received descriptions. We derive asymptotical expressions for $D_{3,k}$ for $k = 1, 2$, where $D_{3,k}$ denotes the average distortion when only $k$ out of $3$ descriptions are received. Our results show that a wide range of distortion pairs $(D_{3,1}, D_{3,2})$ can be achieved even if $N$ and $R$ are fixed. In particular, the proposed IA can achieve distortion pairs $(D_{3,1}, D_{3,2})$ with ratios $D_{3,1}/D_{3,2} = 12N^{\frac{2}{n}(1-\beta)}$, for all $\beta$ in some compact interval $I \subset (0, 1)$, while keeping the product $D_{3,1}D_{3,2}$ the same as in [6] as $n \to \infty$. On the other hand, the MDLVQ scheme of [6] can only achieve one distortion pair with the ratio $D_{3,1}/D_{3,2} = 4$. Additionally, it is worth noting that the MDLVQ system of [10] is able to achieve more distortion pairs by varying the weights assigned to different numbers of descriptions in the optimization of the overall expected distortion. However, the distortion pairs obtained in [10] are only linearly related, i.e., the ratio $D_{3,1}/D_{3,2}$ does not depend on $N$ or $R$. Furthermore, the analytical comparison of the structured IA with UEP shows that the proposed IA beats UEP in certain cases.

A preliminary version of this work appeared in the conference paper [21]. We emphasize that the structured IA for $L = 3$ proposed in the current work is significantly different from that developed in [21] and has a better performance. Notably, the IA in the present work achieves the same distortion product as in [6] as $N$ and the vector dimension $n$ approach $\infty$, while the counterpart in [21] exhibits a small gap versus [6]. The present work also includes an extensive empirical/analytical performance comparison with the UEP scheme, which does not appear in [21].

The paper is structured as follows. The following section introduces the necessary definitions and notations and reviews the relevant results from the previous work on MDLVQ. Section III introduces the proposed flexible MDLVQ for $L \geq 3$ and the heuristic IA algorithm. The following section presents the empirical performance evaluation of the proposed MDLVQ and the comparison with UEP. A structured IA for the case $L = 3$ and the derivation of the corresponding expressions of the distortions at high resolution, are presented in Section V. This section also includes the analytical comparison with UEP. Finally, Section VI concludes the paper.

## II. PRELIMINARIES

### A. Definitions and Notations

Let $\nu_c$, respectively $\nu_s$, denote the volume of the fundamental region of the central, respectively side lattice. Then $N = \frac{\nu_s}{\nu_c}$. Let $\mathbf{0}$ denote the $n$-dimensional all-zero vector. For a lattice $\Lambda \subset \mathbb{R}^n$

and a lattice point $\lambda \in \Lambda$, the Voronoi region of $\Lambda$ around $\lambda$, is defined as follows

$$V(\lambda) \triangleq \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x} - \lambda\| \leq \|\mathbf{x} - \lambda'\|, \quad \forall \lambda' \in \Lambda\},$$

where $\|\mathbf{y}\| \triangleq \sqrt{\langle \mathbf{y}, \mathbf{y} \rangle}$, with $\langle \mathbf{x}, \mathbf{y} \rangle \triangleq \sum_{i=1}^{n} x_i y_i$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. We will use the subscript $c$, respectively $s$, for the Voronoi region of $\Lambda_c$, respectively $\Lambda_s$.

To represent a missing description at the decoder we use the symbol $*$. For instance, for the case $L = 3$, if the triple $(\lambda_1, \lambda_2, \lambda_3)$ is transmitted, and the second description is lost, then we say that the triple $(\lambda_1, *, \lambda_3)$ is received. We use $L$-bit sequences $\mathbf{b} = b_1 b_2 \cdots b_L \in \{0,1\}^L$ to represent various patterns of received descriptions, where $b_i = 1$ means that description $i$ is received, while $b_i = 0$ means that description $i$ is lost. For $\lambda_c \in \Lambda_c$ and $\mathbf{b} \in \{0,1\}^L$ let us denote by $\eta_{\mathbf{b}}(\lambda_c)$ the reconstruction of $\lambda_c$ when the pattern of received descriptions is $\mathbf{b}$. For $\mathbf{b} \in \{0,1\}^L$, let $H(\mathbf{b})$ denote the Hamming weight of $\mathbf{b}$, which equals the number of 1's in $\mathbf{b}$. Therefore, $H(\mathbf{b})$ equals the number of received descriptions corresponding to pattern $\mathbf{b}$. The per symbol squared error is used as a distortion measure. Thus, the distortion of the source reconstruction corresponding to pattern $\mathbf{b}$ is

$$D_{\mathbf{b}} = \frac{1}{n} \sum_{\lambda_c \in \Lambda_c} \int_{V_c(\lambda_c)} \|\mathbf{x} - \eta_{\mathbf{b}}(\lambda_c)\|^2 \mathbf{f}(\mathbf{x}) \mathbf{dx},$$

where $\mathbf{f}(\mathbf{x}) = \prod_{j=1}^{n} f(x_j)$ is the $n$-fold probability density function (pdf) of the source vectors $\mathbf{x}$. Further, the distortion when $k$ descriptions out of $L$ are received, $1 \leq k \leq L$, is defined as

$$D_{L,k} \triangleq \frac{1}{\binom{L}{k}} \sum_{\substack{\mathbf{b} \in \{0,1\}^L \\ H(\mathbf{b}) = k}} D_{\mathbf{b}}. \tag{1}$$

Notice that $D_{L,L}$ equals the distortion of the lattice quantizer with codebook $\Lambda_c$, referred to as the central distortion and denoted by $D_c$.

As in [1], [6] we will assume that the IA $\alpha$ satisfies the *shift-invariance property*, i.e., that

$$\alpha_i(\lambda_c + u) = \alpha_i(\lambda_c) + u, \forall \lambda_c \in \Lambda_c, u \in \Lambda_s, 1 \leq i \leq L.$$

Additionally, since $\alpha$ is shift-invariant we will assume that the decoder mapping is shift invariant as well, i.e.,

$$\eta_{\mathbf{b}}(\lambda_c + u) = \eta_{\mathbf{b}}(\lambda_c) + u, \forall \lambda_c \in \Lambda_c, u \in \Lambda_s, \mathbf{b} \in \{0,1\}^L.$$

As in the previous work on MDLVQ we will assume that the source is smooth and has a finite differential entropy. Additionally, we will use the high resolution assumption in order to derive the analytical expressions of the distortions. In other words, we will derive an approximation of each distortion, which becomes accurate as $\nu_c \to 0$ and $\nu_s \to 0$. Thus, we assume that $\nu_c$ is small enough so that the pdf of the source vectors is approximately uniform over each Voronoi region of $\Lambda_c$. This implies that each central lattice point is approximately equal to the centroid of its Voronoi region. Then [1], [5], [6]

$$D_{\mathbf{b}} \approx D_c + \frac{1}{n} \sum_{\lambda_c \in \Lambda_c} \|\lambda_c - \eta_{\mathbf{b}}(\lambda_c)\|^2 \int_{V_c(\lambda_c)} \mathbf{f}(\mathbf{x}) \mathbf{dx},$$

for every pattern $\mathbf{b}$ of received descriptions with $H(\mathbf{b}) < L$. Based on the above relation, on the shift-invariance of the IA and of the decoding mapping, and on the assumption that the pdf of the source vectors is uniform over each Voronoi cell of the side lattice, one obtains [1], [5], [6]

$$D_{\mathbf{b}} \approx D_c + \frac{1}{nN} \sum_{\lambda_c \in V_s(\mathbf{0})} \|\lambda_c - \eta_{\mathbf{b}}(\lambda_c)\|^2.$$

Using further (1), it follows that

$$D_{L,k} \approx D_c + \frac{1}{nN} \frac{1}{\binom{L}{k}} \sum_{\lambda_c \in V_s(\mathbf{0})} \sum_{\substack{\mathbf{b} \in \{0,1\}^L \\ H(\mathbf{b})=k}} \|\lambda_c - \eta_{\mathbf{b}}(\lambda_c)\|^2. \tag{2}$$

As in the previous work on MDLVQ we assume that entropy coding is used to encode each side description. It can be easily verified that the shift invariance property of the IA ensures that $|\alpha_i^{-1}(\lambda_s)| = N$, for every $1 \le i \le L$, $\lambda_s \in \Lambda_s$, where $|\mathcal{B}|$ denotes the cardinality of the set $\mathcal{B}$. Based on the above observation and on the assumption that the pdf of the source vectors is uniform over each set $\alpha_i^{-1}(\lambda_s)$, $1 \le i \le L$, $\lambda_s \in \Lambda_s$, it follows that each description has the rate $R$ (measured in bits per sample) satisfying [1], [5]

$$R \approx h(f) - \frac{1}{n} \log_2(N\nu_c), \tag{3}$$

where $h(f)$ denotes the differential entropy of $f$, i.e., $h(f) = -\int_{\mathbb{R}} f(x) \log_2 f(x) dx$. Let $R_c$ denote the entropy of the central quantizer. Then, as $\nu_c$ approaches $0$, one has [22]

$$R_c \approx h(f) - \frac{1}{n} \log_2(\nu_c). \tag{4}$$

It is clear that $R$ and $R_c$ have to satisfy the following conditions

$$R \le R_c \le LR. \tag{5}$$

Let us write now as in [8]

$$R_c = R(1 + \alpha(L-1)). \tag{6}$$

Then conditions (5) are equivalent to $0 \leq \alpha \leq 1$. Recall that we require that $\nu_c \to 0$ and $\nu_s \to 0$, which in view of (3) and (4) imply that both $R$ and $R_c$ approach $\infty$. Clearly, as long as relation (6) is valid for some $\alpha$, $0 \leq \alpha \leq 1$, then $R \to \infty$ implies that $R_c \to \infty$. Thus, the only requirement that we need to impose is that $R \to \infty$. Additionally, notice that as $R$ varies $\alpha$ does not have to be fixed. In particular, we may have $\alpha \to 0$.

Note that equation (3) implies that

$$N\nu_c \approx 2^{n(h(f)-R)}. \tag{7}$$

Further, relations (4), (6) and (7) lead to

$$N \approx 2^{n\alpha(L-1)R}. \tag{8}$$

*Remark 1:* Recall that the derivation of (3) relies on the assumption that the pdf of the source vectors is approximately uniform on each set $\alpha_i^{-1}(\lambda_s)$. A sufficient condition for the latter to hold as $R \to \infty$, is that the volume of the convex closure of the set $\cup_{\lambda_c \in \alpha_i^{-1}(\lambda_s)} V_c(\lambda_c)$ to approach 0 for all $i$ and $\lambda_s$. If we assume that the largest such volume equals $N\nu_c \times N^\gamma$, for some real value $\gamma$, then, in view of (7) and (8), the condition $N\nu_c \times N^\gamma \to 0$ holds if and only if

$$R\left(1 - (L-1)\alpha\gamma\right) \to \infty. \tag{9}$$

Since $D_{L,L} = D_c$, according to [23] one has

$$D_{L,L} \approx G(\Lambda_c)\nu_c^{\frac{2}{n}},$$

where $G(\Lambda_c)$ denotes the dimensionless normalized second moment of the lattice $\Lambda_c$, i.e.,

$$G(\Lambda_c) \triangleq \frac{1}{n\nu_c^{\frac{n+2}{n}}} \int_{V_c(\mathbf{0})} \|x\|^2 dx.$$

Then equations (4) and (8) further imply that

$$D_{L,L} \approx G(\Lambda_c) 2^{2(h(f)-R(1+\alpha(L-1))}. \tag{10}$$

### B. Previous MDLVQ Scheme for $L \geq 3$

For every $\lambda_c \in \Lambda_c$, denote

$$\mu_s(\lambda_c) = \frac{1}{L}\sum_{i=1}^{L}\alpha_i(\lambda_c).$$

Additionally, for any $L$-tuple $(\lambda_1, \cdots, \lambda_L) \in \Lambda_s^L$ define its *spread* as follows [5]

$$sp(\lambda_1, \cdots, \lambda_L) = \sum_{i=1}^{L}\left\|\lambda_i - \frac{1}{L}\sum_{j=1}^{L}\lambda_j\right\|^2.$$

Then the decoding rule used in prior work implies that for all $k, 1 \leq k \leq L-1$, one has [5], [6]

$$\begin{aligned} D_{L,k} \quad \approx \quad & D_c + \frac{1}{nN}\sum_{\lambda_c \in V_s(\mathbf{0})}\|\lambda_c - \mu_s(\lambda_c)\|^2 + \\ & \frac{1}{nN}\frac{L-k}{Lk(L-1)}\sum_{\lambda_c \in V_s(\mathbf{0})} sp(\alpha(\lambda_c)). \end{aligned} \tag{11}$$

A key factor in the IA design proposed in [6] is the so-called $L$-fraction lattice $\Lambda_{s/L}$, defined as follows

$$\Lambda_{s/L} \triangleq \frac{1}{L}\Lambda_s = \left\{\lambda_{s/L} \in \mathbb{R}^n : \lambda_{s/L} = \frac{\mathbf{k}}{L}G_s, \mathbf{k} \in \mathbb{Z}^n\right\},$$

where $G_s$ is the generator matrix of $\Lambda_s$. Let us denote by $V_{s/L}(\lambda_{s/L})$ the Voronoi region of $\Lambda_{s/L}$ around $\lambda_{s/L} \in \Lambda_{s/L}$. We will assume that the sublattice $\Lambda_s$ of $\Lambda_c$ is clean, in other words, any central lattice point is contained in a unique Voronoi region of $\Lambda_s$. It is shown in [6] that, if $\Lambda_s$ is clean then any central lattice point is contained in a unique Voronoi region of $\Lambda_{s/L}$ as well. Further, for each $\lambda_{s/L} \in \Lambda_{s/L}$ consider

$$\mathcal{T}\left(\lambda_{s/L}\right) \triangleq \left\{(\lambda_1, \lambda_2, ..., \lambda_L) \in \Lambda_s^L : \frac{1}{L}\sum_{j=1}^{L}\lambda_j = \lambda_{s/L}\right\}.$$

The authors of [6] prove that the procedure of assigning the $L$-tuples in $\mathcal{T}(\lambda_{s/L})$ of smallest spread to the central lattice points in $V_{s/L}\left(\lambda_{s/L}\right)$, minimizes $D_{L,k}$ of (11) for all $k, 1 \leq k \leq L-1$, as $N \to \infty$.

Additionally, the asymptotic analysis in [6], [8] leads, for $1 \leq k \leq L-1$, to

$$D_{L,k} \approx \frac{L-k}{k}L^{-\frac{L}{L-1}}(N\nu_c)^{\frac{2}{n}}G(S_{Ln-n})N^{\frac{2}{(L-1)n}} \tag{12}$$

as $N \to \infty$, where $G\left(S_m\right)$ is the normalized second moment of a sphere in $\mathbb{R}^m$, for $m \geq 1$.

## III. Proposed Flexible MDLVQ Framework for $L \geq 3$

Notice that in the previous MDLVQ schemes once $\Lambda_c$ and $\Lambda_s$ have been chosen the system's performance is determined. Interestingly, the IA used in evaluating the performance minimizes the distortion (as $N \to \infty$) for every possible $k, 1 \leq k \leq L-1$. The fact that it is possible to optimize simultaneously the distortions for all values of $k$ is contrary to our intuition that there should be tradeoffs between these distortions. The first insight towards solving this conflict is that the optimality of the IA is a result of the particular way the decoder is designed. The second insight is that the decoding rule is natural for the side decoders, i.e., when $k = 1$, while for $2 \leq k \leq L-1$, it seems to be rather dictated by convenience. These observations lead to the conclusion that distortion $D_{L,1}$ is, indeed, the smallest possible, but there should be room for further decreasing $D_{L,k}$, for $2 \leq k \leq L-1$, if $D_{L,1}$ is allowed to increase. Nevertheless, for this to be possible, the decoder mappings for $2 \leq k \leq L-1$ have to be changed.

Therefore, in order to introduce more flexibility into the system we start by considering a different decoding mapping for $2 \leq k \leq L-1$, as follows. For any $L$-tuple $(\xi_1, \xi_2, \cdots, \xi_L) \in \left(\Lambda_s \cup \{*\}\right)^L$ that has at least one component equal to $*$ and at least two other components in $\Lambda_s$, we compute the reconstruction value as the arithmetic average of the central lattice points in the set $\alpha^{-1}(\xi_1, \xi_2, \cdots, \xi_L)$, where

$$\alpha^{-1}(\xi_1, \xi_2, \cdots, \xi_L) \triangleq \{\lambda_c \in \Lambda_c : \alpha_i(\lambda_c) = \xi_i \text{ for all } 1 \leq i \leq L, \text{ with } \xi_i \in \Lambda_s\}.$$

Notice that this decoding rule is optimal assuming that the pdf of the source vectors is uniform over the set $\alpha^{-1}(\xi_1, \xi_2, \cdots, \xi_L)$. Further, in order to control the performance when $k$, descriptions are received, for $2 \leq k \leq L-1$, we introduce $L-2$ parameters: $\delta_k$, $2 \leq k \leq L-1$, satisfying the inequalities: $\delta_2 \geq \delta_3 \geq \cdots \geq \delta_{L-1} \geq 0$, and impose the following condition.

**Condition A.** For any $k, 2 \leq k \leq L-1$, and any $L$-tuple $(\xi_1, \xi_2, \cdots, \xi_L) \in \left(\Lambda_s \cup \{*\}\right)^L$ having exactly $L-k$ components equal to $*$, one must have $\|\lambda_c - \lambda_c'\| \leq \delta_k$ for any $\lambda_c, \lambda_c' \in \alpha^{-1}(\xi_1, \xi_2, \cdots, \xi_L)$.

It can be easily verified that Condition A guarantees that

$$D_{L,k} \lessapprox D_c + \delta_k^2, \quad \forall k, 2 \leq k \leq L-1.$$

Finally, when designing the IA $\alpha$ we attempt to minimize $D_{L,1}$ while ensuring that Condition A is satisfied. Note that the decoding rule for $k = 1$ is the same as in the previous work, therefore the value of $D_{L,1}$ can be still computed using equation (11), leading to

$$D_{L,1} \approx D_c + \frac{1}{nN} \sum_{\lambda_c \in V_s(\mathbf{0})} \|\lambda_c - \mu_s(\lambda_c)\|^2 + \frac{1}{nNL} \sum_{\lambda_c \in V_s(\mathbf{0})} sp(\alpha(\lambda_c)). \quad (13)$$

The proposed IA algorithm first selects a set $\mathcal{F}$ of $L^n$ $L$-fraction lattice points $\lambda_{s/L} \in \Lambda_{s/L} \cap V_s(\mathbf{0})$, such that the difference of any two such points is not in $\Lambda_s$. Let us define $\mathcal{C} \triangleq \cup_{\lambda_{s/L} \in \mathcal{F}} \left(V_{s/L}(\lambda_{s/L}) \cap \Lambda_c\right)$. Notice that it is sufficient to specify the IA for the central lattice points in $\mathcal{C}$ and then extend it to $\Lambda_c$ via shifting. Guided by (13), for each $\lambda_{s/L} \in \mathcal{F}$ we assign the central lattice points in $V_{s/L}(\lambda_{s/L})$ to $L$-tuples in $\mathcal{T}(\lambda_{s/L})$, to minimize the first sum in (13). However, we can no longer select the $L$-tuples of smallest spread to be assigned as in [6], since this would lead to violations of Condition A. Having in mind the need to keep the last sum in (13) as small as possible we proceed in a greedy manner as described next.

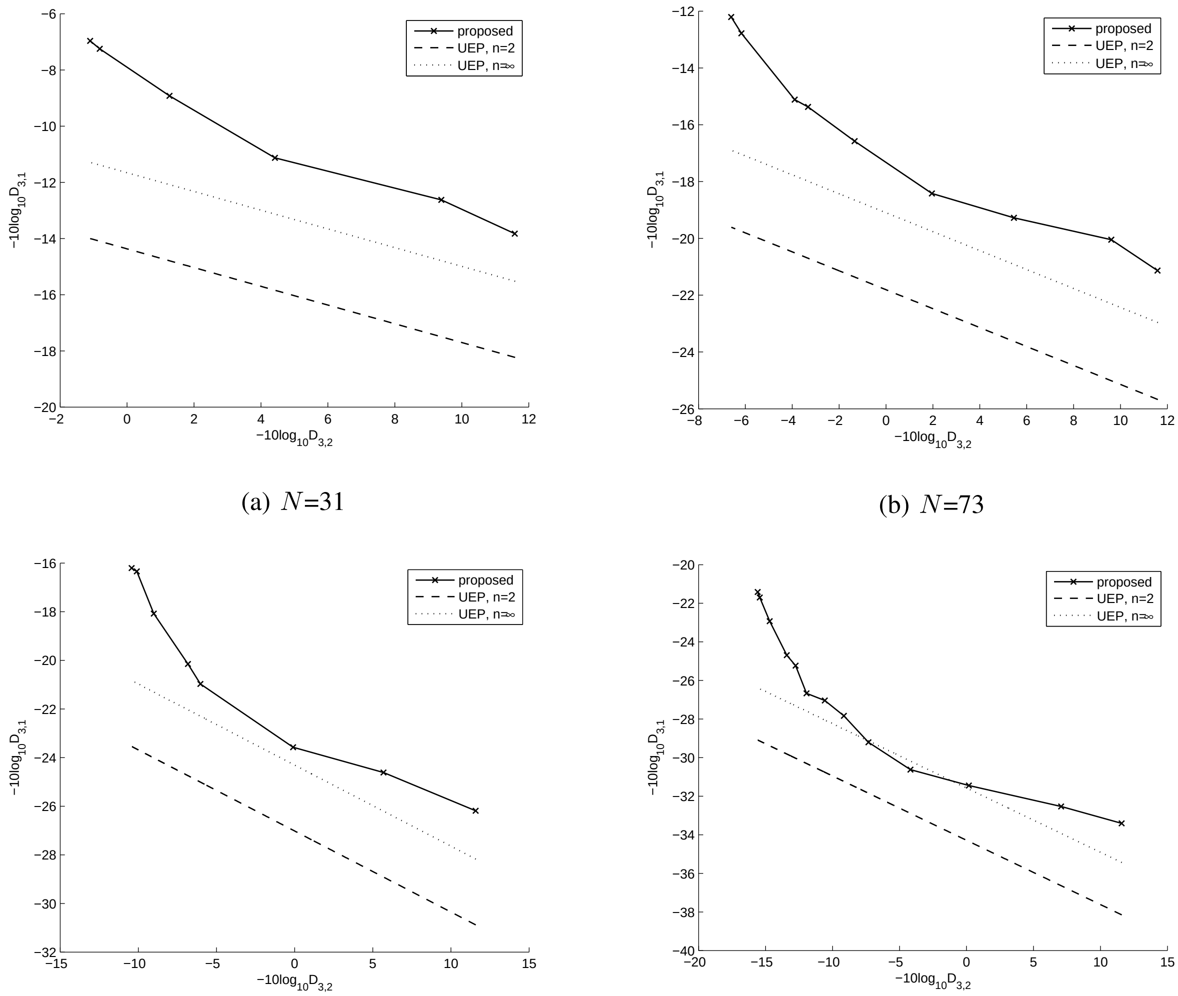
The algorithm maintains a list $\mathcal{T}$ of candidate triples to be assigned. The assignment is built up gradually such that at every moment Condition A to be satisfied for the assignment obtained by extending via shifting the partial assignment built so far. The set $\mathcal{T}$ is initialized to $\mathcal{T} = \cup_{\lambda_{s/L} \in \mathcal{F}} \mathcal{T}(\lambda_{s/L})$. At each iteration the $L$-tuple of smallest spread from the set $\mathcal{T}$ is selected as the current candidate to be assigned. Let $(\lambda_1, \cdots, \lambda_L)$ denote this $L$-tuple. Next the value $\lambda_{s/L} = \frac{\sum_{i=1}^{L} \lambda_i}{L}$ is determined and for each $\lambda_c$ in $V_{s/L}(\lambda_{s/L})$ which has not yet been assigned an $n$-tuple the algorithm tests whether assigning the $L$-tuple to $\lambda_c$ preserves Condition A for the assignment extended by shifting. For this, it is checked whether $\lambda_c$ satisfies Condition B stated next.

**Condition B.** Given $\lambda_c \in \Lambda_c$, we say that $\lambda_c$ satisfies Condition B if and only if for every $\lambda'_c \in \mathcal{C}$ assigned so far and for every $k$, $2 \leq k \leq L-1$, such that there are $s_k \in \Lambda_s$ and $k$ different positions $i_1, i_2, \cdots, i_k \in \{1, \cdots, L\}$ with the property that $\alpha_{i_j}(\lambda'_c) = \alpha_{i_j}(\lambda_c) + s_k$, $1 \leq j \leq k$, the inequality $\|\lambda'_c - s_k - \lambda_c\| \leq \delta_k$ holds.

If a point $\lambda_c$ satisfying Condition B is found then the $L$-tuple is assigned to it and removed from the list $\mathcal{T}$. Otherwise the $L$-tuple is simply removed from the list $\mathcal{T}$ without being assigned. When all central lattice points from the set $V_{s/L}(\lambda_{s/L})$, for some $\lambda_{s/L}$, are assigned, all $L$-tuples in $\mathcal{T}(\lambda_{s/L}) \cap \mathcal{T}$ are removed from $\mathcal{T}$. Finally, the algorithm stops when $\mathcal{T}$ becomes empty.

We point out that when $\delta_k = \infty$ for all $k, 2 \leq k \leq L-1$, the algorithm produces an IA as in

[6].




(a) $N$=31

(b) $N$=73

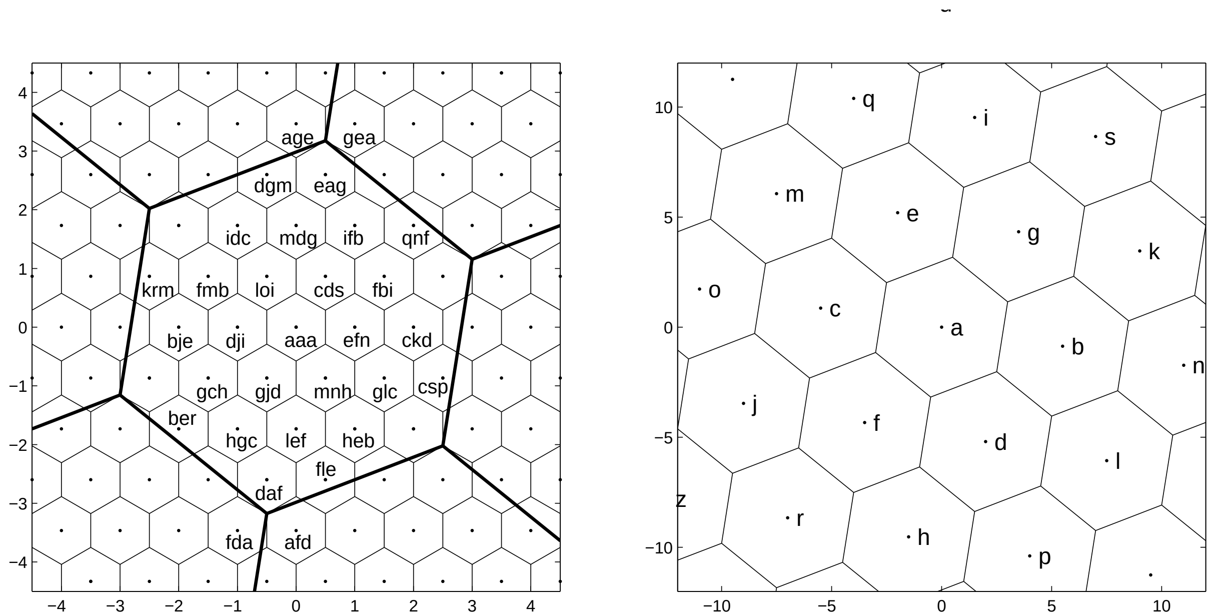
(c) $N$=133

(d) $N$=307

Fig. 1: Performance of the proposed heuristic IA and the comparison with the UEP scheme.

## IV. Experimental Results

In this section, we assess empirically the performance of the proposed IA algorithm for $L = 3$ and $L = 4$. In both cases $n = 2$ and the central lattice $\Lambda_c$ is the hexagonal lattice $A_2$, i.e., the lattice generated by the vectors $(1, 0)$ and $(-1/2, \sqrt{3}/2)$. Thus, $\nu_c = \frac{\sqrt{3}}{2}$. The side lattice $\Lambda_s$ is a clean sublattice of $\Lambda_c$.




(a) (b)

Fig. 2: Three description IA for the $A_2$ lattice with $N = 31$. In (2a) the small hexagons are the Voronoi regions of the central lattice and the big hexagon is the Voronoi region $V_s(\mathbf{0})$. Each lower case letter labels a sublattice point, with "a" labelling $\mathbf{0}$. The sublattice points and their corresponding labels are shown in (2b).

For $L = 3$ we consider the following values for the index $N$, $N = 31, 73, 133, 307$. We use various values for the parameter $\delta_2$, starting at $0$. The step size used to increment $\delta_2$ depends on the particular value of $N$. For each $N$, once $\delta_2$ reaches a particular value no change in performance is observed by further increasing $\delta_2$.

Fig. 1 plots the value of $-10\log_{10} D_{3,1}$ versus $-10\log_{10} D_{3,2}$ for each $N$, and various values of $\delta_2$. The distortions are computed using relation (2), which holds if $R$ is sufficiently large. For this, in view of (3), we assume that $h(f)$ is sufficiently large.

As $\delta_2$ decreases towards $0$, the value of $D_{3,2}$ decreases towards $D_c$, while the value of $D_{3,1}$ increases. The leftmost point in each plot (corresponding to the smallest $D_{3,1}$ and the largest $D_{3,2}$) is achieved for the largest $\delta_2$. Its corresponding pair of distortions $(D_{3,2}, D_{3,1})$ is the same as in [6]. As we see in the figure, the proposed MDLVQ achieves the desired tradeoffs between $D_{3,1}$ and $D_{3,2}$.

In Fig. 1 we have included for comparison the plots of the distortion pairs (in dB) obtained using the UEP scheme for a Gaussian source, with the same description rate $R$ and the same value of $D_{3,3}$ as the proposed scheme. We assume that the variance of the Gaussian source is large enough so that $R$ is sufficiently large.

Recall that the UEP scheme [14], [15] uses a successively refinable source code (SRC) along with unequal erasure protection to obtain the descriptions. To generate $L$ descriptions the bitstream output by the SRC is divided into $L$ consecutive segments called layers. Let $R_k$ denote the rate of the prefix formed out of the first $k$ layers, $1 \leq k \leq L$. The distortion achieved when $k$ descriptions are available equals the distortion obtained when the first $k$ layers of the SRC are decoded, denoted by $D_o(R_k)$. Additionally, the description rate $R_{UEP}$ satisfies the following relation

$$R_{UEP} = \sum_{i=1}^{L-1} \frac{R_i}{i(i+1)} + \frac{R_L}{L}. \tag{14}$$

We consider two different situations for the UEP scheme, labelled in Fig. 1 as $n = 2$, respectively $n = \infty$. The label $n = \infty$ corresponds to the case when the vector dimension used in the SRC approaches $\infty$. Then one has

$$D_{o,n=\infty}(R_k) = \frac{1}{2\pi e} 2^{2(h(f)-R_k)}. \tag{15}$$

Based on equations (14) and (15) and letting $R_{UEP}$ be equal to $R$ from (3) and $D_{o,n=\infty}(R_3) = D_c = G(A_2)\nu_c$, one obtains that the pairs $(D_{3,1}, D_{3,2})$ corresponding the UEP with $n = \infty$ satisfy the relation

$$\frac{1}{12} \log_2 \left(D_{3,1}^3 D_{3,2}\right) = -\frac{1}{6} \log_2 \left(G(A_2)\nu_c\right) + \frac{1}{2} \log_2 \frac{N\nu_c}{2\pi e},$$

which is the basis of the plots for UEP with $n = \infty$ in Fig. 1.

On the other hand, the label $n = 2$ corresponds to the case when the vector dimension used in the SRC is 2. More specifically, we assume that the SRC consists of a sequence of three embedded quantizers, where each quantizers has a scaled version of the $A_2$ lattice as the codebook. Then, under the high resolution assumption, one obtains that

$$D_{o,n=2}(R_k) = G(A_2) 2^{2(h(f)-R_k)}, \tag{16}$$

for $1 \leq k \leq L$, as $R_1 \to \infty$ and $R_k - R_{k-1} \to \infty$ for $1 < k \leq L$. Further, it follows that the pairs $(D_{3,1}, D_{3,2})$ corresponding to UEP with $n = 2$ and the same description rate $R$ and value

TABLE I: $L = 4, N = 31$

| $\delta_2$ | $\delta_3$ | $D_{4,1}$ | $D_{4,2}$ | $D_{4,3}$ | $\Delta R$ |
|---|---|---|---|---|---|
| 0 | 0 | 31.45 | 0.07 | 0.07 | 0.27 |
| 1 | 0 | 24.08 | 0.10 | 0.07 | 0.32 |
| 2 | 0 | 21.33 | 0.22 | 0.07 | 0.27 |
| 3 | 0 | 13.45 | 0.67 | 0.07 | 0.30 |
| 5 | 0 | 8.08 | 1.69 | 0.07 | 0.38 |
| $\infty$ | 0 | 6.83 | 1.92 | 0.07 | 0.42 |
| 4 | 0 | 10.58 | 1.07 | 0.07 | 0.33 |
| 4 | 1 | 7.58 | 1.09 | 0.10 | 0.43 |
| 4 | 2 | 5.58 | 0.96 | 0.29 | 0.49 |
| 4 | 3 | 4.58 | 1.06 | 0.33 | 0.54 |
| 5 | 0 | 8.08 | 1.69 | 0.07 | 0.38 |
| 5 | 1 | 5.70 | 1.38 | 0.10 | 0.50 |
| 5 | 2 | 4.70 | 1.14 | 0.34 | 0.52 |
| 5 | 3 | 3.83 | 1.10 | 0.40 | 0.59 |
| 5 | 4 | 3.58 | 1.10 | 0.40 | 0.62 |
| 6 | 0 | 6.83 | 1.82 | 0.07 | 0.43 |
| 6 | 1 | 5.58 | 1.58 | 0.10 | 0.50 |
| 6 | 2 | 4.58 | 1.23 | 0.34 | 0.52 |
| 6 | 3 | 3.83 | 1.10 | 0.41 | 0.59 |
| 6 | 4 | 3.58 | 1.09 | 0.41 | 0.62 |

TABLE II: $L = 4, N = 73$

| $\delta_2$ | $\delta_3$ | $D_{4,1}$ | $D_{4,2}$ | $D_{4,3}$ | $\Delta R$ |
|---|---|---|---|---|---|
| 0 | 0 | 163.07 | 0.07 | 0.07 | 0.3 |
| 4 | 0 | 91.07 | 0.74 | 0.07 | 0.22 |
| 6 | 0 | 55.82 | 2.11 | 0.07 | 0.27 |
| 8 | 0 | 31.32 | 4.78 | 0.07 | 0.38 |
| $\infty$ | 0 | 21.57 | 6.69 | 0.07 | 0.47 |
| 4 | 3 | 57.82 | 0.85 | 0.28 | 0.28 |
| 6 | 3 | 32.57 | 2.03 | 0.47 | 0.35 |
| 7 | 3 | 19.20 | 2.92 | 0.59 | 0.49 |
| 8 | 3 | 15.07 | 3.55 | 0.68 | 0.54 |
| $\infty$ | 3 | 13.20 | 3.80 | 0.73 | 0.58 |
| 6 | 0 | 55.82 | 2.11 | 0.07 | 0.27 |
| 6 | 1 | 50.95 | 2.01 | 0.09 | 0.29 |
| 6 | 2 | 45.70 | 2.10 | 0.13 | 0.31 |
| 6 | 3 | 32.57 | 2.03 | 0.47 | 0.35 |
| 6 | 4 | 32.32 | 2.09 | 0.55 | 0.34 |
| 8 | 0 | 31.32 | 4.78 | 0.07 | 0.38 |
| 8 | 2 | 19.45 | 4.07 | 0.25 | 0.49 |
| 8 | 3 | 15.07 | 3.55 | 0.68 | 0.54 |
| 8 | 4 | 12.70 | 3.44 | 0.97 | 0.59 |
| 8 | 5 | 10.95 | 3.42 | 1.14 | 0.63 |

TABLE III: $L = 4, N = 133$

| $\delta_2$ | $\delta_3$ | $D_{4,1}$ | $D_{4,2}$ | $D_{4,3}$ | $\Delta R$ |
|---|---|---|---|---|---|
| 0 | 0 | 636.46 | 0.07 | 0.07 | 0.23 |
| 4 | 0 | 334.08 | 0.64 | 0.07 | 0.20 |
| 8 | 0 | 185.71 | 3.78 | 0.07 | 0.20 |
| 12 | 0 | 101.08 | 10.04 | 0.07 | 0.30 |
| $\infty$ | 0 | 49.83 | 16.09 | 0.07 | 0.50 |
| 4 | 4 | 257.83 | 0.84 | 0.33 | 0.17 |
| 8 | 4 | 108.83 | 3.77 | 0.66 | 0.26 |
| 10 | 4 | 75.08 | 5.86 | 0.71 | 0.33 |
| 12 | 4 | 42.96 | 8.43 | 0.97 | 0.47 |
| $\infty$ | 4 | 31.33 | 9.76 | 1.17 | 0.56 |
| 10 | 0 | 151.58 | 6.79 | 0.07 | 0.20 |
| 10 | 2 | 114.58 | 6.44 | 0.14 | 0.27 |
| 10 | 4 | 75.08 | 5.86 | 0.71 | 0.33 |
| 10 | 6 | 52.08 | 6.09 | 1.39 | 0.42 |
| 10 | 8 | 41.70 | 6.69 | 1.93 | 0.47 |
| 12 | 0 | 101.08 | 10.04 | 0.07 | 0.30 |
| 12 | 2 | 74.33 | 9.94 | 0.18 | 0.36 |
| 12 | 4 | 42.96 | 8.43 | 0.97 | 0.47 |
| 12 | 6 | 26.83 | 7.29 | 2.02 | 0.62 |
| 12 | 8 | 23.46 | 7.41 | 2.53 | 0.65 |

of $D_{3,3}$ as the proposed scheme, satisfy the relation

$$\frac{1}{12}\log_2\left(D_{3,1}^3 D_{3,2}\right) = -\frac{1}{6}\log_2\left(G(A_2)\nu_c\right) + \frac{1}{2}\log_2\left(G(A_2)N\nu_c\right).$$

The results in Fig. 1 demonstrate that the proposed MDLVQ with the heuristic IA not only significantly outperforms the UEP with the same vector dimension, but it is also better in almost all cases than the UEP with $n = \infty$.

We point out that we did not compare directly the heuristic IA with the IA of [7], but we can draw some conclusions regarding the comparison based on the results reported in [7]. As the authors of [7] show their IA slightly outperforms the IA of [6]. Therefore, we expect that by applying the algorithm of [7] to obtain a distortion pair slightly better than the pair in Fig. 1 corresponding to the largest $\delta_2$. On the other hand, the IA of [7] still produces only one distortion pair, thus not being able to achieve tradeoffs between $D_{3,1}$ and $D_{3,2}$.

Fig. 2a illustrates the IA obtained with the proposed heuristic algorithm for the case when $n = 2$, $L = 3$, $\Lambda_c = A_2$ and $N = 31$. The small hexagons in the figure represent the Voronoi regions of the central lattice, while the large hexagon is $V_s(\mathbf{0})$. The figure shows the IA for the central lattice points in the set $\mathcal{C}$. Fig. 2b shows the side lattice points and their labels, with "a" labelling the point $\mathbf{0}$.

The results for $L = 4$ are presented in Tables I-III for $N = 31, 73, 133$. From the simulation results, we see that by varying $\delta_2$ and $\delta_3$ we achieve tradeoffs between $D_{4,1}$, $D_{4,2}$ and $D_{4,3}$. The tables also contain the value of $\Delta R$, which is the additional description rate needed by the UEP scheme with $n = 2$, in order to achieve the same distortion quadruple $(D_{4,k})_{1\leq k\leq 4}$ as the proposed scheme. According to (14), (16), (3) and equation $D_{4,4} = G(A_2)\nu_c$, one obtains that

$$\Delta R = \frac{1}{2}\log_2 N + \frac{3}{8}\log_2\left(G(A_2)\nu_c\right) - \frac{1}{24}\log_2\left(D_{4,1}^6 D_{4,2}^2 D_{4,3}\right).$$

The results illustrated in Tables I-III demonstrate that the proposed MDLVQ is far superior to the UEP with the same vector dimension. Furthermore, the difference between the description rate needed by the UEP with $n = \infty$, and the rate of the proposed scheme with $n = 2$, can be computed by subtracting the value $\frac{1}{2}\log_2\left(G(A_2)2\pi e\right) \approx 0.2251$ from $\Delta R$. Thus, we see that in most of the cases the proposed MDLVQ remains superior to the UEP even when the vector dimension of the latter approaches $\infty$.

## V. Structured Index Assignment for $L = 3$

The IA proposed in Section III lacks structure and therefore it is difficult to analyze theoretically. In this section we propose a structured IA for a flexible MDLVQ in the case $L = 3$ and derive its asymptotical performance. For this we assume that $N \to \infty$. In view of relation (8), the above condition is equivalent to $\alpha R \to \infty$, which implies that $R \to \infty$. Notice that for $\alpha R \to \infty$ to hold, $\alpha$ does not need to be fixed, but it can vary in the interval $(0, 1]$ as $R$ varies. We point out that the condition $N \to \infty$ is ubiquitous in all prior work on MDLVQ that derives its analytical performance.

We first develop an IA ensuring $D_{3,2} = D_c$ and then proceed to the case $D_{3,2} > D_c$.

### A. *Case* $D_{3,2} = D_c$

Consider the set $\mathcal{A}_0$ consisting of the $N$ side lattice points that are closest to $\mathbf{0}$. Further, let $\mathcal{T}_0$ denote the set of triples $(\lambda, \mathbf{0}, -\lambda)$ with $\lambda \in \mathcal{A}_0$. The central lattice points in $V_s(\mathbf{0})$ are assigned to the triples in $\mathcal{T}_0$ in a one-to-one manner. Further, the IA is extended via shifting to all points in $\Lambda_c$. Thus, the central lattice points in $V_s(v)$, for $v \in \Lambda_s$ are assigned triples of the form $(\lambda + v, v, -\lambda + v)$. It is easy to see that any two components of any assigned triple $(\lambda_1, \lambda_2, \lambda_3)$

uniquely identify the corresponding central lattice point $\lambda_c = \alpha^{-1}(\lambda_1, \lambda_2, \lambda_3)$. Therefore, one has

$$D_{3,2} = D_c \approx G(\Lambda_c)\nu_c^{\frac{2}{n}}.$$

In order to proceed with the derivation of $D_{3,1}$ we need the following result, which follows from [1].

*Lemma 1:* Let $\Lambda'$ be a lattice in $\mathbb{R}^n$ with $\nu'$ denoting the volume of its fundamental region, and let $N_0$ be a positive integer. Denote by $S(N_0, \Lambda')$ the sum of the squared distances to $\mathbf{0}$ of the $N_0$ lattice points that are closest to $\mathbf{0}$. Additionally, let $r(N_0, \Lambda')$ denote the radius of the smallest sphere centered in $\mathbf{0}$, whose convex closure[1] contains the $N_0$ lattice points that are closest to $\mathbf{0}$. Then one has

$$\begin{aligned} S(N_0, \Lambda') &= N_0 \left(N_0\nu'\right)^{\frac{2}{n}} nG(S_n)(1 + o(1)), \\ r(N_0, \Lambda') &= \left(N_0\nu'\right)^{\frac{1}{n}} \sqrt{(n + 2)G(S_n)}(1 + o(1)), \end{aligned}$$

as[2] $N_0 \to \infty$.

Relation (13) implies that

$$D_{3,1} \approx D_c + \frac{1}{nN} \sum_{\lambda_c \in V_s(\mathbf{0})} \|\lambda_c\|^2 + \frac{2}{3nN} \sum_{\lambda \in \mathcal{A}_0} \|\lambda\|^2. \quad (17)$$

We will first prove that

$$D_c + \frac{1}{nN} \sum_{\lambda_c \in V_s(\mathbf{0})} \|\lambda_c\|^2 \approx G(\Lambda_s)(N\nu_c)^{\frac{2}{n}}. \quad (18)$$

Let $D_s$ denote the distortion of a quantizer having $\Lambda_s$ as a codebook. Then, according to [23], as $\nu_s$ approaches $0$ one has

$$D_s \approx \frac{\int_{V_s(\mathbf{0})} \|\mathbf{x}\|^2 \mathbf{dx}}{nN\nu_c} = G(\Lambda_s)(N\nu_c)^{\frac{2}{n}}.$$

The above relation together with the fact that $\lambda_c$ approximates the centroid of $V_c(\lambda_c)$ further imply that

$$\begin{aligned} D_s &\approx \frac{\sum_{\lambda_c \in V_s(\mathbf{0})} \int_{V_c(\lambda_c)} \|\mathbf{x} - \lambda_c + \lambda_c\|^2 \mathbf{dx}}{nN\nu_c} \\ &\approx \frac{N \int_{V_c(\mathbf{0})} \|\mathbf{x}\|^2 \mathbf{dx}}{nN\nu_c} + \frac{\sum_{\lambda_c \in V_s(\mathbf{0})} \|\lambda_c\|^2}{nN}, \end{aligned}$$

[1]We point out that the convex closure of a sphere equals the union between the sphere and its interior.

[2]For an arbitrary function $f(n)$, relation $f(n) = o(1)$ holds as $n \to \infty$ if and only if $\lim_{n \to \infty} f(n) = 0$.

which proves (18) since [23]

$$D_c \approx \frac{\int_{V_c(\mathbf{0})} \|\mathbf{x}\|^2 \mathbf{dx}}{n\nu_c}.$$

Further, notice that $\sum_{\lambda \in \mathcal{A}_0} \|\lambda\|^2 = S(N, \Lambda_s)$, and based on Lemma 1 and equations (17), (18), one obtains the following

$$\begin{aligned} D_{3,1} &\approx G(\Lambda_s)(N\nu_c)^{\frac{2}{n}} + \frac{2}{3} G\left(S_n\right) N^{\frac{2}{n}} (N\nu_c)^{\frac{2}{n}} \\ &\approx \frac{2}{3} G\left(S_n\right) N^{\frac{2}{n}} (N\nu_c)^{\frac{2}{n}}, \end{aligned} \tag{19}$$

where the last relation is obtained by keeping only the dominant term as $N \to \infty$.

Let us now express the distortions in terms of the rate $R$. Let us first verify if condition (9) is satisfied. Clearly, for $\lambda_s \in \Lambda_s$, the set $\alpha_i^{-1}(\lambda_s)$ is most spread out if $i = 1$ or $i = 3$. It is sufficient to analyze the case $i = 3$. It can be easily seen that the set $\alpha_3^{-1}(\lambda_s)$ contains a central lattice point from each Voronoi region $V_s(v)$, with $v \in \lambda_s + \mathcal{A}_0$, where $\lambda_s + \mathcal{A}_0 \triangleq \{\lambda_s + \lambda : \lambda \in \mathcal{A}_0\}$. Then we may approximate the volume of the convex closure of $\cup_{\lambda_c \in \alpha_i^{-1}(\lambda_s)} V_c(\lambda_c)$ by the volume of the set $\cup_{v \in \lambda_s + \mathcal{A}_0} V_s(v)$, which equals $N\nu_s$. Then the value of $\gamma$ in Remark 1 is $\gamma = 1$, and relation (9) is equivalent to

$$R(1 - 2\alpha) \to \infty. \tag{20}$$

Note that both conditions $\alpha R \to \infty$ and (20) are satisfied if $\alpha$ remains in some compact interval $I \subset (0, 1/2)$, while $R \to \infty$. Additionally, note that (20) implies that $\alpha < \frac{1}{2}$, which is not a restriction, but rather a natural condition for the requirement $D_{3,2} = D_c$ to hold since $2R$ cannot be smaller than $R_c$.

Using further relations (7), (8), (10) and (19) it follows that the following hold

$$D_{3,1} \approx \frac{2}{3} G\left(S_n\right) 2^{2(h(f) - R(1-2\alpha))}, \tag{21}$$

$$D_{3,2} = D_{3,3} \approx G(\Lambda_c) 2^{2(h(f) - R(1+2\alpha))}, \tag{22}$$

for $0 < \alpha < \frac{1}{2}$ and $R \to \infty$ such that $\alpha R \to \infty$ and $R(1 - 2\alpha) \to \infty$.

### B. *Case* $D_{3,2} > D_c$

Let $\mu > 0$ and let $\mathcal{U}$ denote the set of side lattice points in the convex closure of the $n$ dimensional sphere of radius $\mu$, centered at $\mathbf{0}$. Let $\mathcal{A}$ denote the set of the closest $\lceil \frac{N}{|\mathcal{U}|} \rceil$ side lattice points to $\mathbf{0}$. Assume that $1 < |\mathcal{U}| < \sqrt{N}$. Then we can write $|\mathcal{U}| = N^{\frac{\beta}{2}}$ for some

$\beta \in (0,1)$. We require that $|\mathcal{U}| \to \infty$, which is equivalent to the condition that $\alpha\beta R \to \infty$, and which also implies that $N \to \infty$. Notice that for $\alpha\beta R \to \infty$ to hold, $\alpha$ and $\beta$ do not have to be fixed. They may vary as $R$ varies. Additionally, note that, as $|\mathcal{U}| \to \infty$, one has $|\mathcal{A}||\mathcal{U}| \approx N$. Therefore, we will assume that $|\mathcal{A}| = N^{1-\frac{\beta}{2}}$.

Consider now a mapping $\varphi : \mathcal{U} \to \Lambda_s$ which assigns to each $u \in \mathcal{U}$ a side lattice point denoted by $\varphi(u)$ such that $\frac{u}{6} \in V_s(-\varphi(u))$, or, in other words, $\frac{u}{6} + \varphi(u) \in V_s(\mathbf{0})$. Define the set

$$\mathcal{T}(\mathbf{0}) \triangleq \{(\lambda + \varphi(u), \varphi(u), -\lambda + u + \varphi(u)) : \lambda \in \mathcal{A}, u \in \mathcal{U}\}.$$

It can be easily verified that for any two distinct ordered pairs $(\lambda, u), (\lambda', u') \in \Lambda_s^2$, one has

$$(\lambda + \varphi(u), \varphi(u), -\lambda + u + \varphi(u)) \neq$$
$$(\lambda' + \varphi(u'), \varphi(u'), -\lambda' + u' + \varphi(u')).$$

It follows that $|\mathcal{T}(\mathbf{0})| = |\mathcal{A}||\mathcal{U}| = N$. Then the central lattice points in $V_s(\mathbf{0})$ are assigned triples from the set $\mathcal{T}(\mathbf{0})$. To perform the assignment the set $V_s(\mathbf{0}) \cap \Lambda_c$ is first partitioned into $|\mathcal{U}|$ subsets $\mathcal{L}_u$, $u \in \mathcal{U}$, such that $\lfloor \frac{N}{|\mathcal{U}|} \rfloor \leq |\mathcal{L}_u| \leq \lceil \frac{N}{|\mathcal{U}|} \rceil$ and $\mathcal{L}_u$ contains at most one point $\lambda_c$ for which $-\lambda_c$ is not in $\mathcal{L}_u$. A moment of thought reveals that such a partition is possible since $V_s(\mathbf{0}) \cap \Lambda_c$ is symmetric with respect to $\mathbf{0}$. The latter condition ensures that

$$\left\| \sum_{\lambda_c \in \mathcal{L}_u} \lambda_c \right\| \leq \ell, \tag{23}$$

as $N \to \infty$, for each $u \in \mathcal{U}$, where $\ell$ is the covering radius of $\Lambda_s$, i.e., $\ell = \max\{\|\mathbf{x}\| : \mathbf{x} \in V_s(\mathbf{0})\}$. Finally, for each $u \in \mathcal{U}$ the central lattice points in each $\mathcal{L}_u$ are assigned triples of the form $(\lambda + \varphi(u), \varphi(u), -\lambda + u + \varphi(u))$, where $\lambda \in \mathcal{A}$. Further, the IA is extended to $\Lambda_c$ using shifting. It can be easily verified that the IA mapping obtained this way is injective.

Let us now derive $D_{3,2}$. To simplify the analysis we assume the following suboptimal decoding method. For every $(\lambda_1, \lambda_2, \lambda_3) \in \Lambda_s^3$, let $\eta(\lambda_1, \lambda_2, *) \triangleq \lambda_2$, $\eta(*, \lambda_2, \lambda_3) \triangleq \lambda_2$ and $\eta(\lambda_1, *, \lambda_3) \triangleq \frac{\lambda_1 + \lambda_3}{2}$, where $\eta$ denotes the decoding mapping. It is easy to see that this decoder is shift invariant. Then, according to (2), the following holds

$$D_{3,2} \approx D_c + \frac{1}{3nN} \sum_{\lambda_c \in V_s(\mathbf{0})} \left( \|\lambda_c - \eta_{110}(\lambda_c)\|^2 + \|\lambda_c - \eta_{011}(\lambda_c)\|^2 + \|\lambda_c - \eta_{101}(\lambda_c)\|^2 \right). \tag{24}$$

To proceed with the derivation we need the following lemma, which is proved in the appendix.

*Lemma 2:* Let $\mathbf{y}, \mathbf{y}_i \in \mathbb{R}^n$ for $1 \leq i \leq m$. Let $\bar{\mathbf{y}} \triangleq \frac{\sum_{i=1}^m \mathbf{y}_i}{m}$. Then the following relation holds

$$\sum_{i=1}^{m} \|\mathbf{y} - \mathbf{y}_i\|^2 = m\|\mathbf{y} - \bar{\mathbf{y}}\|^2 + \sum_{i=1}^{m} \|\bar{\mathbf{y}} - \mathbf{y}_i\|^2.$$

Recall that for $\lambda_c \in \mathcal{L}_u$, one has $\alpha(\lambda_c) = (\lambda + \varphi(u), \varphi(u), -\lambda + u + \varphi(u))$, for some $\lambda \in \mathcal{A}$. Then $\eta_{110}(\lambda_c) = \eta_{011}(\lambda_c) = \varphi(u)$ and $\eta_{101}(\lambda_c) = \frac{u}{2} + \varphi(u)$. It follows that

$$\frac{\eta_{110}(\lambda_c) + \eta_{011}(\lambda_c) + \eta_{101}(\lambda_c)}{3} = \frac{u}{6} + \varphi(u).$$

Relation (24) and Lemma 2 imply that

$$\begin{aligned}
&D_{3,2} \approx D_c + \\
&\frac{1}{3nN} \sum_{u \in \mathcal{U}} \sum_{\lambda_c \in \mathcal{L}_u} \left( 3\left\|\lambda_c - \left(\frac{u}{6} + \varphi(u)\right)\right\|^2 + \frac{1}{6}\|u\|^2 \right) = \\
&D_c + \frac{1}{3nN} \sum_{u \in \mathcal{U}} \sum_{\lambda_c \in \mathcal{L}_u} \Bigg( 3\|\lambda_c\|^2 + 3\left\|\frac{u}{6} + \varphi(u)\right\|^2 - \\
&\qquad 6\left\langle \lambda_c, \frac{u}{6} + \varphi(u) \right\rangle + \frac{1}{6}\|u\|^2 \Bigg) = \\
&D_c + \frac{1}{nN} \sum_{\lambda_c \in V_s(\mathbf{0})} \|\lambda_c\|^2 + \frac{1}{18nN} \sum_{u \in \mathcal{U}} \sum_{\lambda_c \in \mathcal{L}_u} \|u\|^2 + \\
&\qquad \frac{1}{nN} \sum_{u \in \mathcal{U}} \sum_{\lambda_c \in \mathcal{L}_u} \left\|\frac{u}{6} + \varphi(u)\right\|^2 - \\
&\qquad \frac{2}{nN} \sum_{u \in \mathcal{U}} \left\langle \sum_{\lambda_c \in \mathcal{L}_u} \lambda_c, \frac{u}{6} + \varphi(u) \right\rangle. \qquad (25)
\end{aligned}$$

To rewrite the third term in the last relation in (25) we use Lemma 1 and obtain that

$$\begin{aligned}
&\tfrac{1}{18nN} \sum_{u \in \mathcal{U}} \sum_{\lambda_c \in \mathcal{L}_u} \|u\|^2 = \tfrac{1}{18nN} |\mathcal{A}|\, S(|\mathcal{U}|, \Lambda_s) \approx \\
&\tfrac{1}{18} G(S_n)\, (N\nu_c)^{\frac{2}{n}}\, |\mathcal{U}|^{\frac{2}{n}} = \tfrac{1}{18} G(S_n)\, (N\nu_c)^{\frac{2}{n}}\, N^{\frac{\beta}{n}}. \qquad (26)
\end{aligned}$$

Plugging (18) and (26) in (25) leads to

$$D_{3,2} \approx G(\Lambda_s)\, (N\nu_c)^{\frac{2}{n}} + \frac{1}{18} G(S_n)\, (N\nu_c)^{\frac{2}{n}}\, N^{\frac{\beta}{n}} + T, \qquad (27)$$

where

$$\begin{aligned}
T \triangleq\ & \frac{1}{nN} \sum_{u \in \mathcal{U}} \sum_{\lambda_c \in \mathcal{L}_u} \left\|\frac{u}{6} + \varphi(u)\right\|^2 - \\
& \qquad \frac{2}{nN} \sum_{u \in \mathcal{U}} \left\langle \sum_{\lambda_c \in \mathcal{L}_u} \lambda_c, \frac{u}{6} + \varphi(u) \right\rangle.
\end{aligned}$$

Next we will show that the second term dominates in (27) when $|\mathcal{U}| \to \infty$. For this we will derive an upper bound for $T$. Recall that $\frac{u}{6} + \varphi(u) \in V_s(\mathbf{0})$. Therefore, one has

$$\left\| \frac{u}{6} + \varphi(u) \right\| \leq \ell. \tag{28}$$

Further, using the Cauchy-Schwarz inequality along with (23) and (28) one obtains

$$\left| \left\langle \sum_{\lambda_c \in \mathcal{L}_u} \lambda_c, \frac{u}{6} + \varphi(u) \right\rangle \right| \leq \left\| \sum_{\lambda_c \in \mathcal{L}_u} \lambda_c \right\| \left\| \frac{u}{6} + \varphi(u) \right\| \leq \ell^2. \tag{29}$$

Relations (28) and (29) imply that

$$T \leq \frac{|\mathcal{U}||\mathcal{A}|}{nN}\ell^2 + 2\frac{|\mathcal{U}|}{nN}\ell^2 \approx \frac{\ell^2}{n}\left(1 + \frac{2}{N^{1-\frac{\beta}{2}}}\right) = k(\Lambda_s)\left(N\nu_c\right)^{\frac{2}{n}}\left(1 + \frac{2}{N^{1-\frac{\beta}{2}}}\right), \tag{30}$$

where

$$k(\Lambda_s) \triangleq \frac{\ell^2}{n\nu_s^{\frac{2}{n}}} \tag{31}$$

is a constant that does not change as $N$ increases (it does not change if $\Lambda_s$ is scaled). Finally, using (30) we conclude that the second term in (27) is dominant as $|\mathcal{U}| \to \infty$, therefore one obtains

$$D_{3,2} \approx \frac{1}{18}G(S_n)\left(N\nu_c\right)^{\frac{2}{n}} N^{\frac{\beta}{n}}. \tag{32}$$

Let us now evaluate $D_{3,1}$. According to (13) one has

$$D_{3,1} \approx D_c + \frac{1}{nN}\sum_{\lambda_c \in V_s(\mathbf{0})} \left\|\lambda_c - \mu_s(\lambda_c)\right\|^2 + \frac{1}{3nN}\sum_{\lambda_c \in V_s(\mathbf{0})}\sum_{j=1}^{3}\left\|\alpha_j(\lambda_c) - \mu_s(\lambda_c)\right\|^2. \tag{33}$$

We will first evaluate the last summation in (33). Note that for $\lambda_c \in \mathcal{L}_u$, one has

$$\mu_s(\lambda_c) = \frac{u}{3} + \varphi(u).$$

Then the following equalities hold

$$\frac{1}{3nN}\sum_{\lambda_c\in V_s(\mathbf{0})}\sum_{j=1}^{3}\|\alpha_j(\lambda_c)-\mu_s(\lambda_c)\|^2 =$$
$$\frac{1}{3nN}\sum_{u\in\mathcal{U}}\sum_{\lambda\in\mathcal{A}}\left(\left\|\lambda-\frac{u}{3}\right\|^2+\left\|\frac{u}{3}\right\|^2+\left\|\lambda-\frac{2u}{3}\right\|^2\right)=$$
$$\frac{1}{3nN}\sum_{u\in\mathcal{U}}\sum_{\lambda\in\mathcal{A}}\left(2\|\lambda\|^2+\frac{2}{3}\|u\|^2-2\langle\lambda,u\rangle\right)=$$
$$\frac{1}{3nN}\left(2|\mathcal{U}|\sum_{\lambda\in\mathcal{A}}\|\lambda\|^2+\frac{2}{3}|\mathcal{A}|\sum_{u\in\mathcal{U}}\|u\|^2\right)=$$
$$\frac{1}{3nN}\left(2|\mathcal{U}|S(|\mathcal{A}|,\Lambda_s)+\frac{2}{3}|\mathcal{A}|S(|\mathcal{U}|,\Lambda_s)\right)=$$
$$G(S_n)(N\nu_c)^{\frac{2}{n}}\left(\frac{2}{3}|\mathcal{A}|^{\frac{2}{n}}+\frac{2}{9}|\mathcal{U}|^{\frac{2}{n}}\right)=$$
$$G(S_n)(N\nu_c)^{\frac{2}{n}}\left(\frac{2}{3}N^{\frac{2}{n}(1-\frac{\beta}{2})}+\frac{2}{9}N^{\frac{\beta}{n}}\right), \tag{34}$$

where the second last equality follows from Lemma 1, while the third last equality relies on the fact that $\sum_{u\in\mathcal{U}}\sum_{\lambda\in\mathcal{A}}\langle\lambda,u\rangle=\sum_{\lambda\in\mathcal{A}}\langle\lambda,\sum_{u\in\mathcal{U}}u\rangle=0$ because $\mathcal{U}$ is symmetric with respect to the origin.

For the remaining portion of (33) we will derive an upper bound as follows.

$$D_c+\frac{1}{nN}\sum_{\lambda_c\in V_s(\mathbf{0})}\|\lambda_c-\mu_s(\lambda_c)\|^2=$$
$$D_c+\frac{1}{nN}\sum_{u\in\mathcal{U}}\sum_{\lambda_c\in\mathcal{L}_u}\left\|\lambda_c-\left(\frac{u}{3}+\varphi(u)\right)\right\|^2=$$
$$D_c+\frac{1}{nN}\sum_{u\in\mathcal{U}}\sum_{\lambda_c\in\mathcal{L}_u}\left(\|\lambda_c\|^2+\left\|\frac{u}{3}+\varphi(u)\right\|^2-\right.$$
$$\left.\left\langle\lambda_c,\frac{u}{3}+\varphi(u)\right\rangle\right)\leq$$
$$D_c+\frac{1}{nN}\left(\sum_{\lambda_c\in V_s(\mathbf{0})}\|\lambda_c\|^2+|\mathcal{A}|\sum_{u\in\mathcal{U}}\left\|\frac{u}{3}+\varphi(u)\right\|^2+\right.$$
$$\left.2\sum_{u\in\mathcal{U}}\left|\left\langle\sum_{\lambda_c\in\mathcal{L}_u}\lambda_c,\frac{u}{3}+\varphi(u)\right\rangle\right|\right) \tag{35}$$

Using the inequality $\|\mathbf{y}_1+\mathbf{y}_2\|^2\leq 2\left(\|\mathbf{y}_1\|^2+\|\mathbf{y}_2\|^2\right)$ and (28) one gets

$$\left\|\frac{u}{3}+\varphi(u)\right\|^2=\left\|\frac{u}{6}+\left(\frac{u}{6}+\varphi(u)\right)\right\|^2\leq 2\left\|\frac{u}{6}\right\|^2+2\left\|\frac{u}{6}+\varphi(u)\right\|^2\leq 2\left(\left\|\frac{u}{6}\right\|^2+\ell^2\right). \tag{36}$$

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Applying further (23), (36) and the inequality $2|\langle \mathbf{y}_1, \mathbf{y}_2\rangle| \leq \|\mathbf{y}_1\|^2 + \|\mathbf{y}_2\|^2$, for $\mathbf{y}_1, \mathbf{y}_2 \in \mathbb{R}^n$, one obtains that

$$2\left|\left\langle \sum_{\lambda_c\in\mathcal{L}_u} \lambda_c, \frac{u}{3} + \varphi\left(u\right)\right\rangle\right| \leq \left\|\sum_{\lambda_c\in\mathcal{L}_u} \lambda_c\right\|^2 + \left\|\frac{u}{3} + \varphi(u)\right\|^2 \leq 2\left\|\frac{u}{6}\right\|^2 + 3\ell^2. \tag{37}$$

Relations (35), (36) and (37) imply that

$$\begin{aligned}
&D_c + \frac{1}{nN}\sum_{\lambda_c\in V_s(\mathbf{0})} \|\lambda_c - \mu_s\left(\lambda_c\right)\|^2 \leq \\
&D_c + \frac{1}{nN}\sum_{\lambda_c\in V_s(\mathbf{0})} \|\lambda_c\|^2 + \frac{|\mathcal{A}|+1}{18nN} S(|\mathcal{U}|, \Lambda_s) + \\
&\quad \frac{\ell^2}{nN}\left(2|\mathcal{A}||\mathcal{U}| + 3|\mathcal{U}|\right) \approx \\
&(N\nu_c)^{\frac{2}{n}}\left(G(\Lambda_s) + \frac{G(S_n)}{18}|\mathcal{U}|^{\frac{2}{n}} + \frac{G(S_n)}{18|\mathcal{A}|}|\mathcal{U}|^{\frac{2}{n}} + \right.\\
&\quad \left. k(\Lambda_s)\left(2 + \frac{3}{|\mathcal{A}|}\right)\right) \approx \\
&\frac{G(S_n)}{18}(N\nu_c)^{\frac{2}{n}}|\mathcal{U}|^{\frac{2}{n}} = \frac{G(S_n)}{18}(N\nu_c)^{\frac{2}{n}} N^{\frac{\beta}{n}},
\end{aligned}$$

where the third last relation follows from Lemma 1 and (31), while the second last relation is obtained by keeping only the dominant term as $|\mathcal{U}| \to \infty$. Corroborating with (33) and (34) and keeping only the dominant term in the expression of $D_{3,1}$ leads to the conclusion that when $\alpha, \beta \in (0, 1)$ and $\alpha\beta R \to \infty$, the following holds

$$D_{3,1} \approx \frac{2}{3} G(S_n)(N\nu_c)^{\frac{2}{n}} N^{\frac{2}{n}(1-\frac{\beta}{2})}\left(1 + \frac{5}{18} N^{-\frac{2}{n}(1-\beta)}\right). \tag{38}$$

Furthermore, if we impose the additional condition that $\alpha(1-\beta)R \to \infty$, so that $N^{-\frac{2}{n}(1-\beta)} \to 0$, we obtain

$$D_{3,1} \approx \frac{2}{3} G(S_n)(N\nu_c)^{\frac{2}{n}} N^{\frac{2}{n}(1-\frac{\beta}{2})}. \tag{39}$$

Let $I$ be some compact interval included in $(0, 1)$. Then for $N$ sufficiently large relations (32) and (39) hold for all $\beta \in I$. Interestingly, as $\beta$ varies in $I$, the product $D_{3,1}D_{3,2}$ is constant, i.e., it does not depend on $\beta$, achieving

$$D_{3,1}D_{3,2} \approx \frac{1}{27}(G(S_n))^2 (N\nu_c)^{\frac{4}{n}} N^{\frac{2}{n}}. \tag{40}$$

On the other hand, according to (12), the distortion product for the MDLVQ in [6] is

$$D_{3,1}D_{3,2} \approx \frac{1}{27}(G(S_{2n}))^2(N\nu_c)^{\frac{4}{n}}N^{\frac{2}{n}}. \tag{41}$$

It can be seen from (40) and (41) that the proposed IA achieves the same distortion product $D_{3,1}D_{3,2}$ as in [6], as $n$ goes to $\infty$, since $\lim_{n\to\infty} G(S_{2n}) = \lim_{n\to\infty} G(S_n) = \frac{1}{2\pi e}$. However, the MDLVQ scheme of [6] can only achieve one distortion pair $(D_{3,1}, D_{3,2})$ with the ratio $D_{3,1}/D_{3,2} = 4$, while the proposed scheme can achieve a wide range of pairs $(D_{3,1}, D_{3,2})$ with ratios $D_{3,1}/D_{3,2} = 12N^{\frac{2}{n}(1-\beta)}$, for all $\beta \in I$.

Let us now express the distortions in terms of the rate $R$. First let us see when (9) is satisfied. For this we need to determine the value of $\gamma$ from Remark 1. It is easy to see that for fixed $\lambda_s \in \Lambda$, the set $\alpha_i^{-1}(\lambda_s)$ is most spread out for $i = 3$. Notice that the set $\alpha_3^{-1}(\lambda_s)$ contains central lattice points from each set $V_s(v)$, with $v = \lambda_s + \lambda - u - \varphi(u)$, for some $\lambda \in \mathcal{A}$ and $u \in \mathcal{U}$. Recall that $\frac{u}{6} + \varphi(u) \in V_s(\mathbf{0})$. It follows that, as $N \to \infty$, one has $\alpha_3^{-1}(\lambda_s) \subset \cup_{v\in\lambda_s+\mathcal{A}-\mathcal{U}} V_s(v)$, where $\mathcal{A} - \mathcal{U} \triangleq \{\lambda - u : \lambda \in \mathcal{A}, u \in \mathcal{U}\}$. Notice that the set $\cup_{v\in\lambda_s+\mathcal{A}-\mathcal{U}} V_s(v)$ is included in a sphere of radius

$$r = r(|\mathcal{A}|, \Lambda_s) + r(|\mathcal{U}|, \Lambda_s) + \ell.$$

Using further Lemma 1 and (31), the volume of the $n$-dimensional sphere of radius $r$ given above becomes

$$\begin{aligned} vol &= \left(\frac{r}{\sqrt{(n+2)G(S_n)}}\right)^n \approx \\ &\nu_s \left(N^{\frac{1}{n}(1-\frac{\beta}{2})} + N^{\frac{1}{n}\frac{\beta}{2}} + \sqrt{\frac{nk(\Lambda_s)}{(n+2)G(S_n)}}\right)^n = \\ &\nu_s N^{1-\frac{\beta}{2}} \left(1 + N^{\frac{1-\beta}{n}} + \sqrt{\frac{nk(\Lambda_s)}{(n+2)G(S_n)}} N^{-\frac{1}{n}(1-\frac{\beta}{2})}\right)^n. \end{aligned}$$

When $\alpha(1-\beta)R \to \infty$ one has $vol \approx \nu_s N^{1-\frac{\beta}{2}}$. According to the above considerations, the volume of the convex closure of $\cup_{\lambda_c\in\alpha_3^{-1}(\lambda_s)} V_c(\lambda_c)$ is smaller than or equal to $vol$. Then the value of $\gamma$ in Remark 1 satisfies $\gamma \le 1 - \frac{\beta}{2}$, and a sufficient condition for relation (9) to hold is

$$R\left(1 - \alpha(2-\beta)\right) \to \infty. \tag{42}$$

By replacing $N\nu_c$, respectively $N$, from (7), respectively (8), in (39) and in (32) one obtains that, for $0 < \beta < 1$, $0 < \alpha < \frac{1}{2-\beta}$, and $R$ such that relations $\alpha\beta R \to \infty$, $\alpha(1-\beta)R \to \infty$ and

(42) hold, the following are satisfied

$$D_{3,1} \approx \frac{2}{3} G(S_n) 2^{2\left(h(f)-R\left(1-2\alpha\left(1-\frac{\beta}{2}\right)\right)\right)}, \tag{43}$$

$$D_{3,2} \approx \frac{1}{18} G(S_n) 2^{2(h(f)-R(1-\alpha\beta))}. \tag{44}$$

### *C. Comparison with UEP*

In this subsection we compare the performance of the proposed structured IA with the UEP scheme for a Gaussian source. In both cases we assume that the block dimension of the source code approaches $\infty$. Let us denote by $R_{UEP}$ the description rate needed by the UEP scheme to achieve the same distortion values as the proposed MDLVQ with the structured IA, i.e., such that $D_o(R_k) = D_{3,k}$ for $1 \leq k \leq 3$. Then

$$R_k = h(f) - \frac{1}{2} \log_2(2\pi e D_{3,k}), 1 \leq k \leq 3,$$

and

$$R_{UEP} = \frac{R_1}{2} + \frac{R_2}{6} + \frac{R_3}{3} = h(f) - \frac{1}{2} \log_2(2\pi e) - \frac{1}{12} \log_2(D_{3,1}^3 D_{3,2} D_{3,3}^2). \tag{45}$$

Consider first the IA introduced in subsection V-A. According to [24] there is a sequence of lattices $\Lambda_n \subset \mathbb{R}^n$ such that $\lim_{n\to\infty} G(\Lambda_n) = \frac{1}{2\pi e}$. Then applying this result in (21), (22), one obtains that, for $n \to \infty$,

$$\begin{aligned} D_{3,1} &\approx \frac{2}{3} \frac{1}{2\pi e} 2^{2(h(f)-R(1-2\alpha))}, \\ D_{3,2} = D_{3,3} &\approx \frac{1}{2\pi e} 2^{2(h(f)-R(1+2\alpha))}. \end{aligned}$$

Applying further (45), leads to

$$R_{UEP} = R + \frac{1}{4} \log_2 \frac{3}{2} \approx R + 0.1462,$$

results which demonstrates that the proposed MDLVQ with the structured IA for the case $D_{3,2} = D_c$ is strictly better than the UEP scheme.

Consider now the IA in subsection V-B. Recall that under the conditions $0 < \beta < 1$, $0 < \alpha < \frac{1}{2-\beta}$, and $R \to \infty$ such that $\alpha\beta R \to \infty$, $\alpha(1-\beta)R \to \infty$, $(1-\alpha(2-\beta))\, R \to \infty$, relations (43), (44) and (10) hold. Further, by letting $n \to \infty$ and using (45) one obtains that

$$R_{UEP} = R - R\frac{\alpha(1-\beta)}{3} + \frac{1}{12} \log_2 \frac{3^5}{4} \approx R - R\frac{\alpha(1-\beta)}{3} + 0.4937. \tag{46}$$

Clearly, as $\alpha(1-\beta)R \to \infty$, one has $R_{UEP} < R$, which means that the UEP performance is better. On the other hand, relation (46) suggests that the proposed scheme may outperform UEP if $R\frac{\alpha(1-\beta)}{3} < 0.4937$. However, the latter relation contradicts the requirement that $\alpha(1-\beta)R \to \infty$. Therefore, let us turn our attention to the performance analysis in the case when $\alpha(1-\beta)R$ equals some constant value $c$ while $\alpha\beta R \to \infty$ and (42) hold. Under these conditions equations (44) and (10) still hold, while (38) implies that

$$D_{3,1} \approx \frac{2}{3}\left(1+\frac{5}{18}2^{-4c}\right) G(S_n) 2^{2\left(h(f)-R\left(1-2\alpha\left(1-\frac{\beta}{2}\right)\right)\right)}. \tag{47}$$

By letting $n \to \infty$ it further follows that

$$R_{UEP} - R \approx 0.4937 - \frac{c}{3} - \frac{1}{4}\log_2\left(1+\frac{5}{18}2^{-4c}\right). \tag{48}$$

The value of $R_{UEP} - R$ in (48) is positive for $c \in (0, 1.45)$, reaching values up to $0.4$. We conclude that the structured IA proposed in subsection V-B outperforms the UEP scheme when $\alpha R \to \infty$, $(1-\alpha)R \to \infty$ and $\beta \to 1$, while $\alpha(1-\beta)R = c \in (0, 1.45]$. Notice that condition $(1-\alpha)R \to \infty$ is needed in order for (42) to hold. Additionally, notice that the above conditions are satisfied, for instance, when $\alpha$ is fixed in $(0,1)$, $R \to \infty$ and $\beta = 1 - \frac{c}{R}$.

## VI. Conclusion

In the previous work on multiple description lattice vector quantizers (MDLVQ) for $L \geq 3$ descriptions, once the central and side lattice are fixed, it is not possible to adjust the decoding quality when the number of received descriptions is higher than $1$, but lower than $L$. This work proposes a flexible MDLVQ that overcomes the aforementioned shortcoming. For this a different reconstruction method is adopted and a heuristic index assignment algorithm is developed, which uses $L-2$ parameters to control the distortions when $k$ descriptions are received, for $2 \leq k \leq L-1$. Extensive simulations show that the proposed technique achieves the desired tradeoffs and, additionally, is significantly superior to the MD scheme based on unequal erasure protection.

Furthermore, a structured index assignment, amenable to theoretical analysis, is proposed for the case $L = 3$. By deriving the asymptotical expressions of the distortions at high resolution, we show that a wide range of values can be achieved for the distortions when $k = 1$ and $k = 2$. Notably, the product of distortions for $k = 1, 2$ is the same as in the previous work. Future research efforts will be directed to further improving the performance of the flexible MDLVQ and designing structured index assignments for general values of $L$.

## APPENDIX A
## PROOF OF LEMMA 2

The following sequence of relations hold

$$\sum_{i=1}^{m} \|\mathbf{y} - \mathbf{y}_i\|^2 =$$
$$\sum_{i=1}^{m} \|\mathbf{y} - \bar{\mathbf{y}} + \bar{\mathbf{y}} - \mathbf{y}_i\|^2 =$$
$$\sum_{i=1}^{m} \|\mathbf{y} - \bar{\mathbf{y}}\|^2 + \sum_{i=1}^{m} \|\bar{\mathbf{y}} - \mathbf{y}_i\|^2 + 2\sum_{i=1}^{m} \langle \mathbf{y} - \bar{\mathbf{y}}, \bar{\mathbf{y}} - \mathbf{y}_i \rangle =$$
$$m\|\mathbf{y} - \bar{\mathbf{y}}\|^2 + \sum_{i=1}^{m} \|\bar{\mathbf{y}} - \mathbf{y}_i\|^2 + 2\left\langle \mathbf{y} - \bar{\mathbf{y}}, m\bar{\mathbf{y}} - \sum_{i=1}^{m} \mathbf{y}_i \right\rangle.$$

The definition of $\bar{\mathbf{y}}$ implies that the last term in the above expression is $\mathbf{0}$. Thus, the conclusion of Lemma 2 follows.

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