Logic Design

Chapter 2: Introduction to Logic Circuits

Introduction

- Logic circuits operate on digital signals
- Unlike continuous analog signals that have an infinite number of possible values, digital signals are restricted to a few discrete values
- In particular for binary logic circuits, signals can have only two values: 0 and 1.

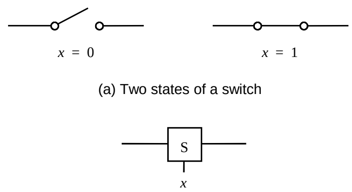
Logic Operations

The *fundamental* logic operations are:

- AND   F = X·Y
- OR   F = X + Y
- NOT   F = X′ (complement)

X′ and $\overline{X}$ are used interchangeably!

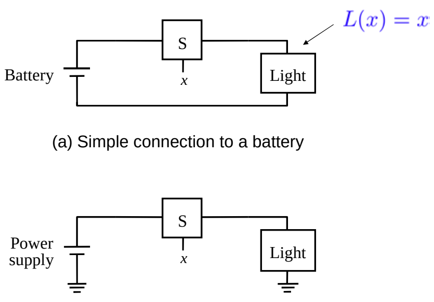
Switch networks



(a) Two states of a switch

(b) Symbol for a switch

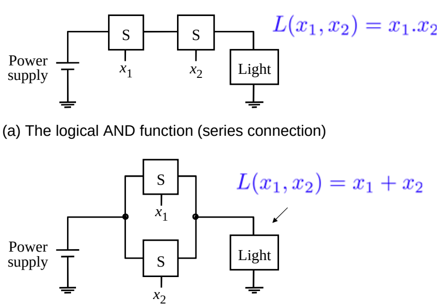
Switch networks



(a) Simple connection to a battery

(b) Using a ground connection as the return path

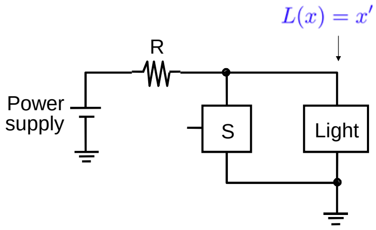
Switch networks



(a) The logical AND function (series connection)

(b) The logical OR function (parallel connection)

Switch networks



Logic Operations

- Don't confuse the AND symbol "·" and OR symbol "+" with arithmetic multiplication and addition
  - There are some differences, for example
    - Arithmetic addition: 1+1=2
    - OR operation: 1+1=1
- Based on the context you should recognize if it is AND/OR or addition/multiplication
- One more thing: sometimes we drop the "·" symbol
  - *a*·*b* is the same as *ab*

Truth table

- The most basic representation of a logic function is a truth table.
- A truth table lists the output of the circuit for every possible input combination.
- There are 2^n rows in a truth table for an n-variable function

Truth table:

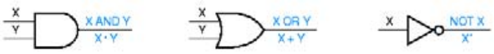
X	Y	XY	X + Y	X′
0	0	0	0	1
0	1	0	1	1
1	0	0	1	0
1	1	1	1	0

Logic Gate

- Binary signals are manipulated using *logic gates*. These are electronic devices whose inputs and outputs are interpreted with only two values, representing logic 0 and logic 1.

Logic Gate

- The bubble on the inverter output denotes "inverting" behavior



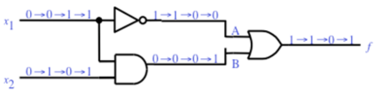
X	Y	X AND Y
0	0	0
0	1	0
1	0	0
1	1	1

X	Y	X OR Y
0	0	0
0	1	1
1	0	1
1	1	1

X	NOT X
0	1
1	0

Analysis and Synthesis of a Logic Network

- Combinations of gates form a logic circuit or logic network
- Analysis: For an existing network determine the function performed by the network
- Synthesis: Design a network that implements a desired function

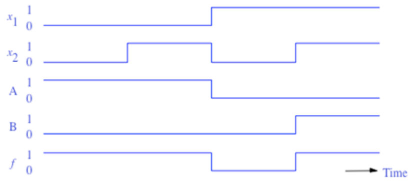


(a) Network that implements $f = \bar{x}_1 + x_1 \cdot x_2$

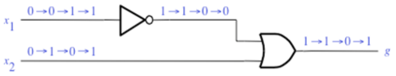
x_1	x_2	$f(x_1, x_2)$
0	0	1
0	1	1
1	0	0
1	1	1

A	B
1	0
1	0
0	0
0	1

(b) Truth table



(c) Timing diagram



(d) Network that implements $g = \bar{x}_1 + x_2$

Boolean Algebra

- To design logic circuits and describe their operations we use a mathematical tool called Boolean algebra (from English mathematician George Boole in 1800's)
  - Boolean algebra operates on two-valued (or logic) functions.
- Key problem of our study:
  - A logic function can be implemented in many ways with logic circuits, what is and how to find the best implementation?

Boolean Algebra

- To design logic circuits and describe their operation requires a mathematical tool called *Boolean algebra* (from English mathematician George Boole in 1800's) that operates on two-valued functions.

Axioms of Boolean algebra

- The axioms (or postulates) of a mathematical system are a minimal set of basic definitions that we assume to be true.
- The first three pairs of axioms state the formal definitions of the AND (logical multiplication) and OR (logical addition) operations:
  (1a) $0 \cdot 0 = 0$ (1b) $1+1 = 1$
  (2a) $1 \cdot 1 = 1$ (2b) $0+0 = 0$
  (3a) $0 \cdot 1 = 1 \cdot 0 = 0$ (3b) $1+0 = 0+1 = 1$
- The next axioms embody the complement notation:
  (4a) If X=0, then X'=1 (4b) If X=1, then X'=0

Theorems of Boolean algebra

- Theorems are statements, known to be true, that allow us to manipulate algebraic expressions to have simpler analysis or more efficient synthesis of the corresponding circuits.
- Theorems involving a single variable:
  (5a) X·0 = 0 (5b) X+1 = 1 (Null elements)
  (6a) X·1 = X (6b) X+0 = X (Identities)
  (7a) X·X = X (7b) X+X = X (Idempotency)
  (8a) X·X' = 0 (8b) X+X' = 1 (Complements)
  (9) (X')' = X (Involution)
- These theorems can be proved to be true. Let us prove 6b:
  [X=0] 0+0=0 (true, according to 2b)
  [X=1] 1+0=1 (true, according to 3b)

Theorems of Boolean algebra

- Theorems involving two or three variables:
  (10a) X·Y = Y·X (10b) X+Y = Y+X (Commutativity)
  (11a) (X·Y)·Z = X·(Y·Z) (11b) (X+Y)+Z = X+(Y+Z) (Associativity)
  (12a) X·Y+X·Z = X·(Y+Z) (12b) (X+Y)·(X+Z) = X+Y·Z (Distributivity)
  (13a) X+X·Y = X (13b) X·(X+Y) = X (Absorption)
  (14a) X·Y+X·Y' = X (14b) (X+Y)·(X+Y') = X (Combining)
  (15a) $(X_1 \cdot X_2)' = X_1' + X_2'$
  (15b) $(X_1 + X_2)' = X_1' \cdot X_2'$ DeMorgan's theorems
  (16a) X+X'·Y= X+Y (16b) X·(X'+Y)= X·Y (simplification)
  (17a) X·Y+X'·Z+Y·Z = X·Y+X'·Z (Consensus)
  (17b) (X+Y)·(X'+Z)·(Y+Z) = (X+Y)·(X'+Z)

Duality

- Theorems were presented in pairs.
- The b version of a theorem is obtained from the a version by swapping "0" and "1", and "·" and "+".
- Principle of Duality: Any theorem or identity in Boolean algebra remains true if 0 and 1 are swapped and · and + are swapped throughout.
- Duality doubles the utilities of everything about Boolean algebra and enriches the manipulation of logic functions.

Consensus theorem

Consensus Theorem:

- $XY + \bar{X}Z + YZ = XY + \bar{X}Z$
  ↑ redundant (YZ)

Note: Y and Z are associated with X and $\bar{X}$, and appear together in the term that is eliminated.

By duality:

$$(x+y)\bullet(y+z)\bullet(\bar{x}+z) = (x+y)\bullet(\bar{x}+z)$$

Boolean Algebra

$X + 0 = X$	$X \cdot 1 = X$	Identity
$X + 1 = 1$	$X \cdot 0 = 0$	
$X + X = X$	$X \cdot X = X$	Idempotent Law
$X + X' = 1$	$X \cdot X' = 0$	Complement
$(X')' = X$		Involution Law
$X + Y = Y + X$	$XY = YX$	Commutativity
$X + (Y + Z) = (X + Y) + Z$	$X(YZ) = (XY)Z$	Associativity
$X(Y + Z) = XY + XZ$	$X + YZ = (X + Y)(X + Z)$	Distributivity
$X + XY = X$	$X(X + Y) = X$	Absorption Law
$X + X'Y = X + Y$	$X(X' + Y) = XY$	Simplification
$(X + Y)' = X'Y'$	$(XY)' = X' + Y'$	DeMorgan's Law
$XY + X'Z + YZ = XY + X'Z$	$(X + Y)(X' + Z)(Y + Z) = (X + Y)(X' + Z)$	Consensus Theorem

Differences between Boolean and ordinary algebra

- Distributive law of + over ·
  x+(y·z)=(x+y)·(x+z) is not valid in ordinary algebra
- Boolean algebra does not have additive or multiplicative inverse so there is no subtraction or division operations

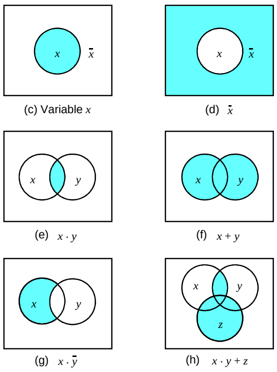
Boolean Algebra

- Boolean algebra is used for manipulating logical functions when designing digital hardware.
- However, today most design is done using Computer-Aided Design (CAD) software that includes schematic capture, logic simplification and simulation.
- Other methods include truth tables, Venn diagrams and Karnaugh Maps.

Venn Diagram

- A graphical tool that can be used for Boolean algebra
- A binary variable s is represented by a contour
- Area within the contour corresponds to s=1
- Area outside the contour corresponds to s=0
- Two variables are represented by two overlapping circles

Venn Diagram



Venn Diagram

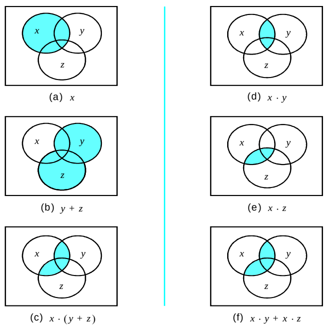


Figure 2.13. Verification of the distributive property
$x \cdot (y + z) = x \cdot y + x \cdot z$

Precedence of operations

- In the absence of parentheses, operations in a logic expression must be performed in the order: NOT, AND, OR.

Example:

$$f = x_1.x_2 + \bar{x}_1\bar{x}_2$$

Synthesis using AND, OR and NOT

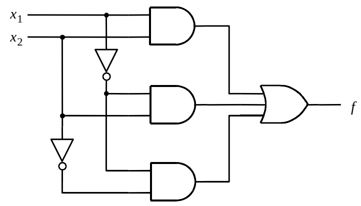
- One way of designing a logic circuit that implements a truth table is to create a product term that has a value of 1 for each valuation for which the output function has to be 1.
- Then we take the logical sum of these product terms to realize f

x_1	x_2	$f(x_1, x_2)$
0	0	1
0	1	1
1	0	0
1	1	1

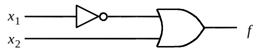
$$f(x_1, x_2) = \overline{x_1}\,\overline{x_2} + \overline{x_1}x_2 + x_1x_2$$

$$f = \overline{x_1} + x_2$$

x_1	x_2	$f(x_1, x_2)$
0	0	1
0	1	1
1	0	0
1	1	1



(a) Canonical sum-of-products



(b) Minimal-cost realization

Minterm, Maxterm

- Minterm
  A product term in which all variables of a function appear exactly once, uncomplemented or complemented.
- Maxterm
  A sum term in which all variables of a function appear exactly once, uncomplemented or complemented.

Minterm, Maxterm

For a Boolean function of n variables, there are 2^n minterms:

$$\mathbf{m_0 \,..\, m_{2^n-1}}$$

and 2^n maxterms:

$$\mathbf{M_0 \,..\, M_{2^n-1}}$$

Note that: $\mathbf{M_i = \overline{m_i}}$

Minterm, Maxterm

Row No.	A B C	Minterms	Maxterms
0	0 0 0	$A'B'C' = m_0$	$A+B+C = M_0$
1	0 0 1	$A'B'C = m_1$	$A+B+C' = M_1$
2	0 1 0	$A'BC' = m_2$	$A+B'+C = M_2$
3	0 1 1	$A'BC = m_3$	$A+B'+C' = M_3$
4	1 0 0	$AB'C' = m_4$	$A'+B+C = M_4$
5	1 0 1	$AB'C = m_5$	$A'+B+C' = M_5$
6	1 1 0	$ABC' = m_6$	$A'+B'+C = M_6$
7	1 1 1	$ABC = m_7$	$A'+B'+C' = M_7$

$$M_i = m_i'$$

Canonical Sum of Products Form

- A Boolean function f(x1,x2,x3) can be expressed algebraically as a logical *sum of minterms*:

Row number	x_1	x_2	x_3	$f(x_1, x_2, x_3)$
0	0	0	0	0
1	0	0	1	1
2	0	1	0	0
3	0	1	1	0
4	1	0	0	1
5	1	0	1	1
6	1	1	0	1
7	1	1	1	0

Canonical Sum of Products Form

- f can be expressed as sum of product terms (SOP)

$$f(x1,x2,x3) = \sum(m1, m4, m5, m6)$$
$$f(x1,x2,x3) = \sum m(1,4,5,6)$$

Canonical Product of Sums Form

- The *complement* of f(x1,x2,x3) can be formed as the logical sum of all minterms not used in f(x1,x2,x3):

$$\bar{f}(x1,x2,x3) = m0 + m2 + m3 + m7$$
$$f = \overline{m0 + m2 + m3 + m7}$$
$$f = \overline{m0} \bullet \overline{m2} \bullet \overline{m3} \bullet \overline{m7}$$
$$f = M0 \bullet M2 \bullet M3 \bullet M7$$

This is called the product of sum presentation of f

Conversion Between the Canonical Forms

- It is easy to convert from one canonical form to other one, simply use the DeMorgan's theorem.

Example:

$$F(A,B,C) = \sum(1,4,5,6,7)$$
$$F'(A,B,C) = \sum(0,2,3)$$
$$F(A,B,C) = (m0 + m2 + m3)' = m_0' m_2' m_3' = M_0 M_2 M_3$$
$$F(A,B,C) = \prod(0,2,3)$$

Cost of a Logic Circuit

- Cost of a logic circuit: total number of gates plus total number of inputs to all gates in the circuit
- The canonical SOP and POS implementations described before are not necessarily minimum cost
- We can simplify them to obtain minimum-cost SOP and POS circuits

Reducing Cost

How can we simplify a logic function?

- There are systematic approached for doing this (e.g., Karnaugh map) that we will learn later
- The other way is to use theorems and properties of Boolean algebra and do algebraic manipulations
- Do an example on the board.

Reducing Cost

- The simplified version of SOP is called minimal SOP
- The simplified version of POS is called minimal POS
- We cannot in general predict whether the minimal SOP expression or minimal POS expression will result in the lowest cost.
- It is often useful to check both expressions to see which gives the best result.

Other Logic Operations

- NAND
- NOR
- XOR
- XNOR

NAND

- NAND: a combination of an AND gate followed by an inverter.

A	B	Y
0	0	1
0	1	1
1	0	1
1	1	0

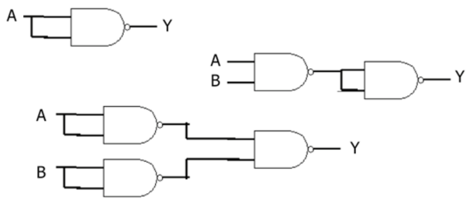
- Symbol for NAND is ↑
- NAND gates have several interesting properties:

$$A \uparrow A = A'$$
$$(A \uparrow B)' = AB$$
$$(A' \uparrow B') = A + B$$

NAND

- These three properties show that a NAND gate with both of its inputs driven by the same signal is equivalent to a NOT gate
- A NAND gate whose output is complemented is equivalent to an AND gate, and a NAND gate with complemented inputs acts as an OR gate.
- Therefore, we can use a NAND gate to implement all three of the *elementary operators* (AND,OR,NOT).
- Therefore, ANY Boolean function can be constructed using only NAND gates.

NAND



NOR

- NOR: a combination of an OR gate followed by an inverter.

A	B	Y
0	0	1
0	1	0
1	0	0
1	1	0

- NOR gates also have several interesting properties:

$$A \downarrow A = A'$$
$$(A \downarrow B)' = A + B$$
$$A' \downarrow B' = AB$$

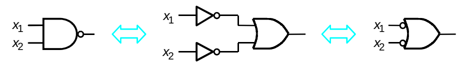
NOR

- Just like the NAND gate, any logic function can be implemented using just NOR gates.
- Both NAND and NOR gates are very valuable as any design can be realized using either one.
- It is easier to build an IC chip using all NAND or NOR gates than to combine AND,OR, and NOT gates.
- NAND/NOR gates are typically faster at switching and cheaper to produce.

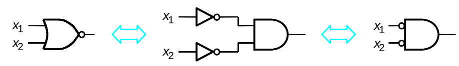
NAND and NOR networks

- NAND and NOR can be implemented by simpler electronic circuits than the AND and OR functions
- Can these gates be used in synthesis of logic circuits?

NAND and NOR networks

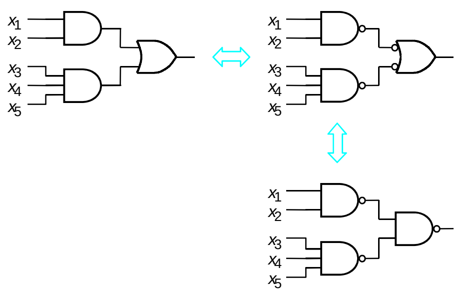


(a) $\overline{x_1 x_2} = \bar{x}_1 + \bar{x}_2$

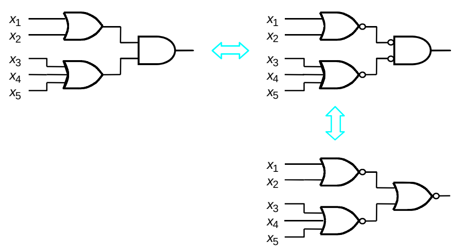


(b) $\overline{x_1 + x_2} = \bar{x}_1 \bar{x}_2$

NAND and NOR networks



NAND and NOR networks



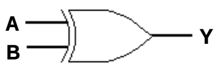
Exclusive OR (XOR)

- The eXclusive OR (XOR) function is an important Boolean function used extensively in logic circuits.
- The XOR function maybe:
  - implemented directly as an electronic circuit (truly a gate) or
  - implemented by interconnecting other gate types (used as a convenient representation)
- The XOR function means:
  X OR Y, but NOT BOTH

XOR

- XOR gates assert their output when exactly one of the inputs is asserted, hence the name.
- The symbol for this operation is ⊕

$Y = A'B + AB'$

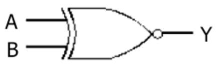


A	B	Y
0	0	0
0	1	1
1	0	1
1	1	0

XNOR

- The eXclusive NOR function is the complement of the XOR function
- The symbol for this operation is ⊙, i.e. 1 ⊙ 1 = 1 and 1 ⊙ 0 = 0.

A	B	Y
0	0	1
0	1	0
1	0	0
1	1	1

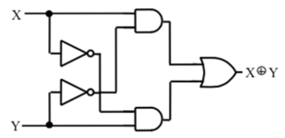


$$Y = A'B' + AB$$

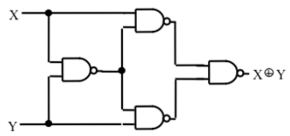
- Why is the XNOR function also known as the *equivalence* function?

XOR Implementations

- A SOP implementation



- A NAND implementation



XOR and XNOR

- Uses for the XOR and XNORs gate include:
  - Adders/subtractors/multipliers
  - Counters/incrementers/decrementers
  - Parity generators/checkers

XOR

- XOR identities:

$$X \oplus 0 = X$$
$$X \oplus X = 0$$
$$X \oplus Y = Y \oplus X$$
$$X \oplus 1 = X'$$
$$X \oplus X' = 1$$

Gates with more than two inputs

- A gate can be extended to have multiple inputs if the binary operation it represents is commutative and associative.
- AND and OR operations have these two properties
- NAND and NOR are not associative:

$$(A \downarrow B) \downarrow C \neq A \downarrow (B \downarrow C)$$
$$(A \uparrow B) \uparrow C \neq A \uparrow (B \uparrow C)$$

Gates with more than two inputs

- We define multiple input NAND and NOR gates as follows:

$$A \downarrow B \downarrow C = (A + B + C)'$$
$$A \uparrow B \uparrow C = (ABC)'$$

Gates with more than two inputs

- XOR and XNOR are both commutative and associative
- Definition of XOR should be modified for more than two inputs
- For more than 2 inputs, XOR is called an *odd function*: it is equal to 1 if the input variables have an odd number of 1's
- Similarly, for more than 2 inputs, XNOR is called an *even function*: it is equal to 1 if the input variables have an even number of 1's

Learning Objectives

- List the three basic logic operations
- Draw the truth table for the basic logic operations
- Build truth table for an arbitrary number of variables
- Draw schematic for basic logic gates
- Perform analysis on simple logic circuits
- Draw timing diagram for simple logic circuits