Graph Algorithms

- Sets and sequences can only model limited relations between objects, e.g. ordering, overlapping, etc.
- Graphs can model more involved relationships, e.g. road and rail networks
- Graph: $G = (V, E)$, V : set of *vertices*, E : set of *edges*
  - Directed graph: an edge is an *ordered* pair of vertices, (v_1, v_2)
  - Undirected graph; an edge is an *unordered* pair of vertices $\{v_1, v_2\}$

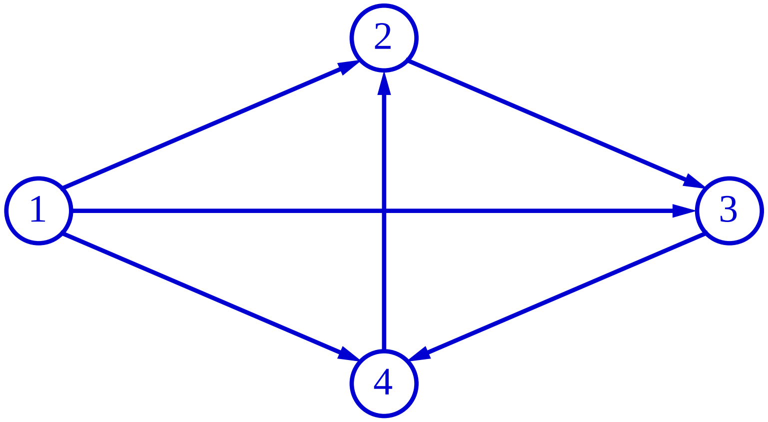
Graph representation

Adjacency matrix

Directed graph

$V = \{1, 2, 3, 4\}$
$E = \{(1, 2), (1, 3), (1, 4), (2, 3), (3, 4), (4, 2)\}$

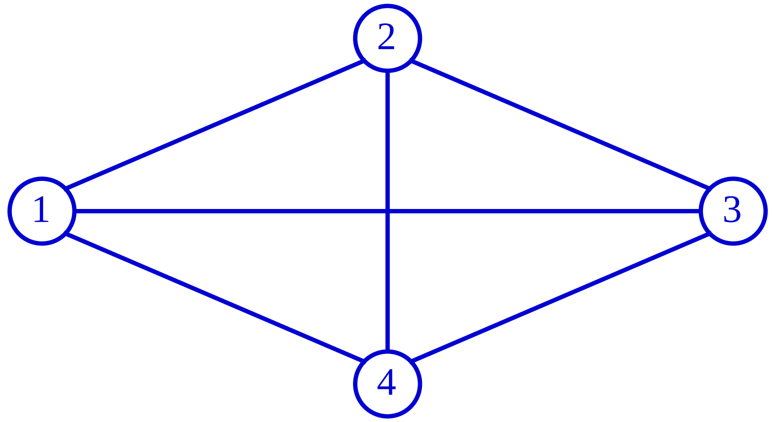
	1	2	3	4
1	0	1	1	1
2	0	0	1	0
3	0	0	0	1
4	0	1	0	0



Undirected graph

$V = \{1, 2, 3, 4\}$
$E = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}$

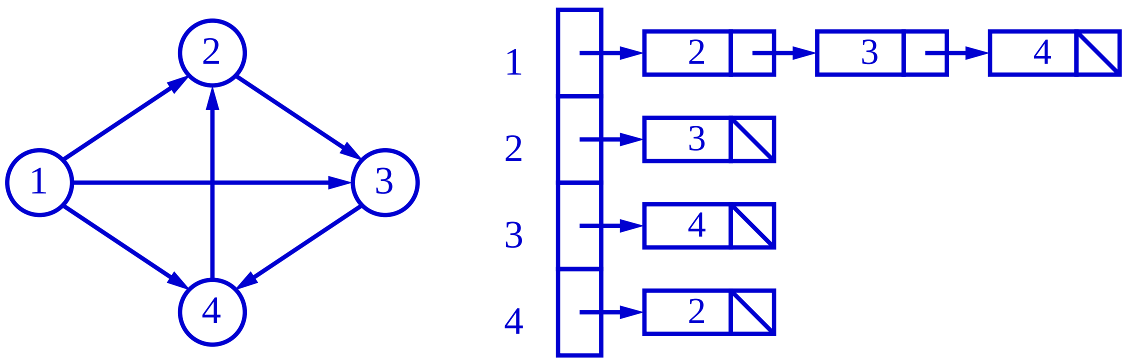
$$\begin{array}{ccccc} & 1 & 2 & 3 & 4 \\ 1 & 0 & 1 & 1 & 1 \\ 2 & 1 & 0 & 1 & 1 \\ 3 & 1 & 1 & 0 & 1 \\ 4 & 1 & 1 & 1 & 0 \end{array} \quad \text{or} \quad \begin{array}{ccccc} & 1 & 2 & 3 & 4 \\ 1 & 0 & 1 & 1 & 1 \\ 2 & & 0 & 1 & 1 \\ 3 & & & 0 & 1 \\ 4 & & & & 0 \end{array}$$



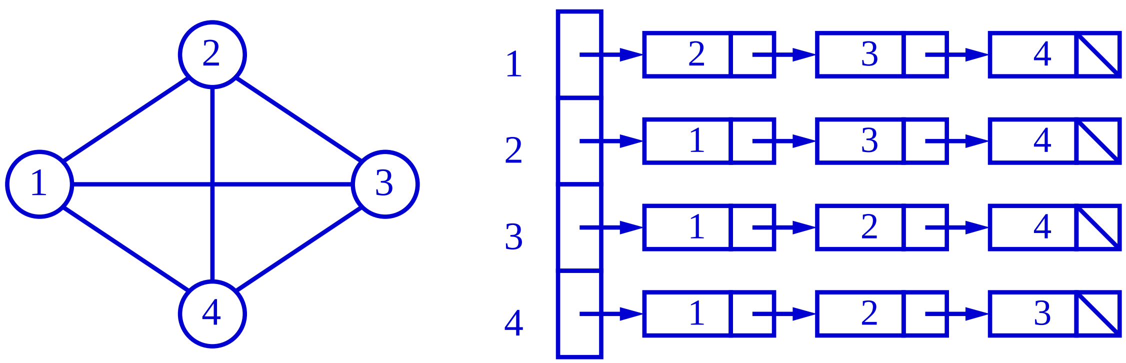
Advantage: $O(1)$ time to check connection.
Disadvantages:
– Space is $O(|V|^2)$ instead of $O(|E|)$
– Finding who a vertex (node) is connected to requires $O(|V|)$ operations

Adjacency List

Example:



Example:



Advantages:

– easy to access all vertices connected to one vertex

– space is $O(|E| + |V|)$

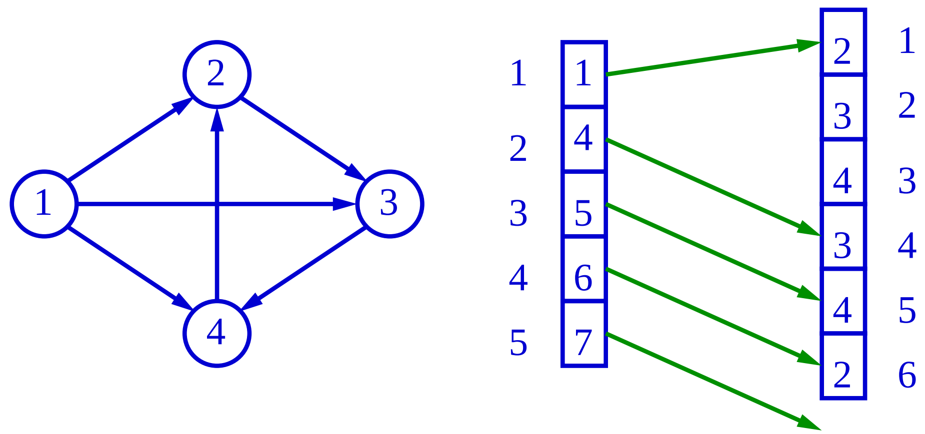
Disadvantage:

– testing connection in worst case is $O(|V|)$

– space: $|V|$ header, $2|E|$ list nodes $\Longrightarrow O(|V| + |E|)$. There might be $|V|^2$ edges ($|E| = |V|^2$) but probably not.

Another representation

Adjacency list with arrays



- For node i, use $header[i]$ and $header[i+1]-1$ as the indices in the *list* array. If $header[i] > header[i+1]-1$ vertex i is not connected to any node.
- same advantage as adjacency list but save space
- binary search is possible to determine the connection: $O(\log|V|)$
- problem: difficult to update the structure

Traversal of a graph

Depth First and **Breadth First**

Depth First (most useful)

var $visited[1 \ldots |V|]$: boolean $\longleftarrow$ $false$

Proc DFS(v);

(Given a graph $G = (V, E)$ and a vertex v, visit each vertex reachable from v)

$visited[v] \longleftarrow true$

perform prework on vertex v

For each vertex w adjacent to v do

    if not $visited[w]$ then

        DFS(w)

        perform postwork on edge (v, w)

    (sometimes we perform postwork on all edges out of v)

– given a vertex v, we need to know all vertices connected to v

– stack space $\approx |V| - 1$

Complexity

1) With adjacency list
visited each vertex once
visited each edge twice; once from v to w, once from w to v.

$O(|V| + |E|)$

2) With adjacency matrix
visited each vertex once
for each vertex, visit all vertices connected to this vertex needs $O(|V|)$ steps

$O(|V|^2)$

Note: In graph, $O(|E|)$ is better than $O(|V|^2)$ in most cases.

Examples

1) DFS numbering

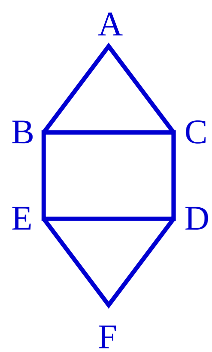
Initially $DFS\_num := 1$

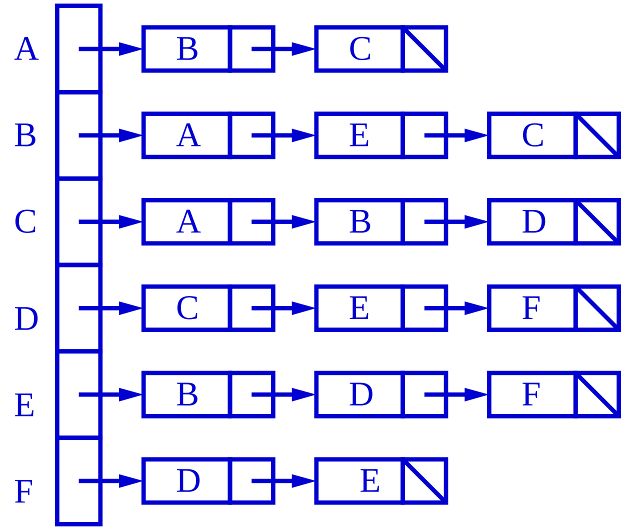
Use DFS with following *prework*

prework

$v.DFS := DFS\_num;$

$DFS\_num := DFS\_num{+}1;$



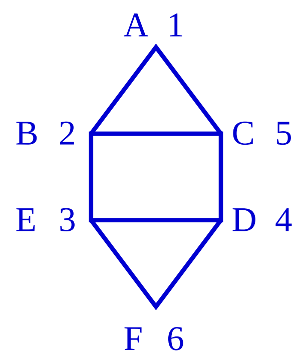


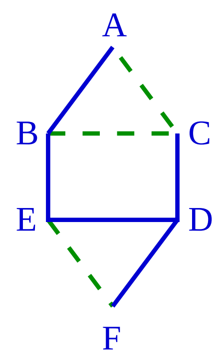
2) DFS tree

Use DFS with following *postwork*

postwork:

add edge (v, w) to T



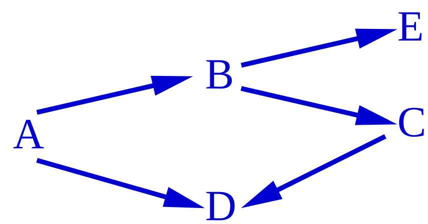


Topological Sorting

Task scheduling

- A set of tasks. Some tasks depend on other tasks
- Task a depends on task b means that task a cannot be started until task b is finished
- We want to find a schedule for tasks consistent with dependencies

Example: $x \rightarrow y$: y cannot start until x is completed.



A B C E D   A B C D E   A B E C D
are all schedule for tasks $\{A, B, C, D, E\}$.

This graph must be acyclic!

The problem

Given a directed acyclic graph $G = (V, E)$ with n vertices, label the vertices from 1 to n such that, if v is labelled k, then all vertices that can be reached from v by a directed path are labelled with labels $> k$.

In other words, label vertices from 1 to n such that for any edge (v, w) the label of v is less than the label of w.

Lemma. A directed acyclic graph always contains a vertex with in-degree 0.
Proof. If all vertices have positive in-degrees, starting from any vertex v, traverse the graph "backward". We never have to stop. But we only have a finite number of vertices!
Consequently, there must be a cycle in the graph – a contradiction! (pigeonhole principle).

Algorithm:

By induction:
find one vertex with in-degree 0. Label this vertex 1, and delete all edges from this vertex to other vertices.
Now the new graph is also acyclic and is of size $n-1$. By induction we know how to label it.

<u>Implementation.</u>

1. Initialize in-degree of all vertices
2. Put all vertices with 0 in-degree into a queue or stack

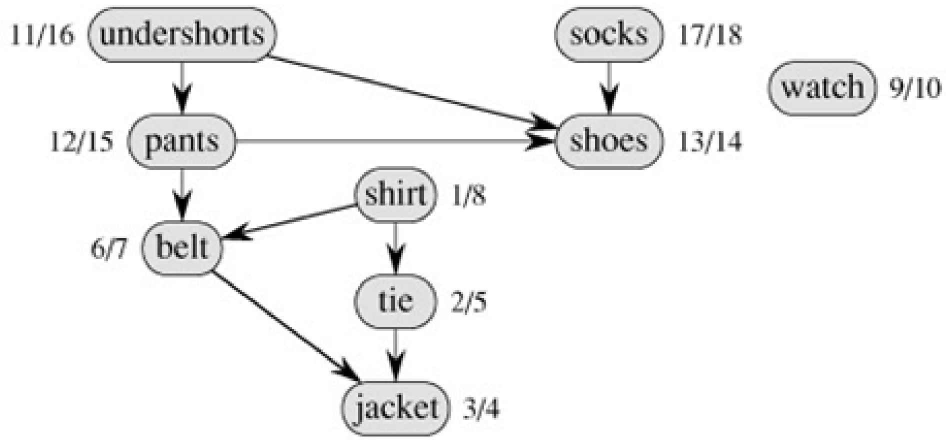
$l \longleftarrow 0$

3. dequeue v; $l \longleftarrow l+1$; $v.label \longleftarrow l$;
   for all edge (v, w)
      decrease in-degree of w by 1
      if degree of w is now 0 enqueue w

until queue is empty

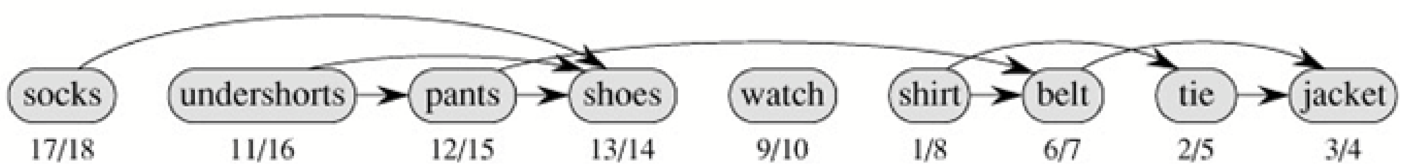
Time: $O(|E|+|V|)$

(a)



undershorts
11/16
socks
17/18
watch
9/10
pants
12/15
shoes
13/14
shirt
1/8
belt
6/7
tie
2/5
jacket
3/4

(b)



socks
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6/7
tie
2/5
jacket
3/4

Single-Source Shortest-Paths

- Weighted graph

  $G = (V, E)$ directed graph with weights associated with the edges
- The weight of an edge (u, v) is $w(u, v)$.

  The weight of a path $p =< v_0, v_1, \cdots v_k >$ is the summation of the weights of its edges

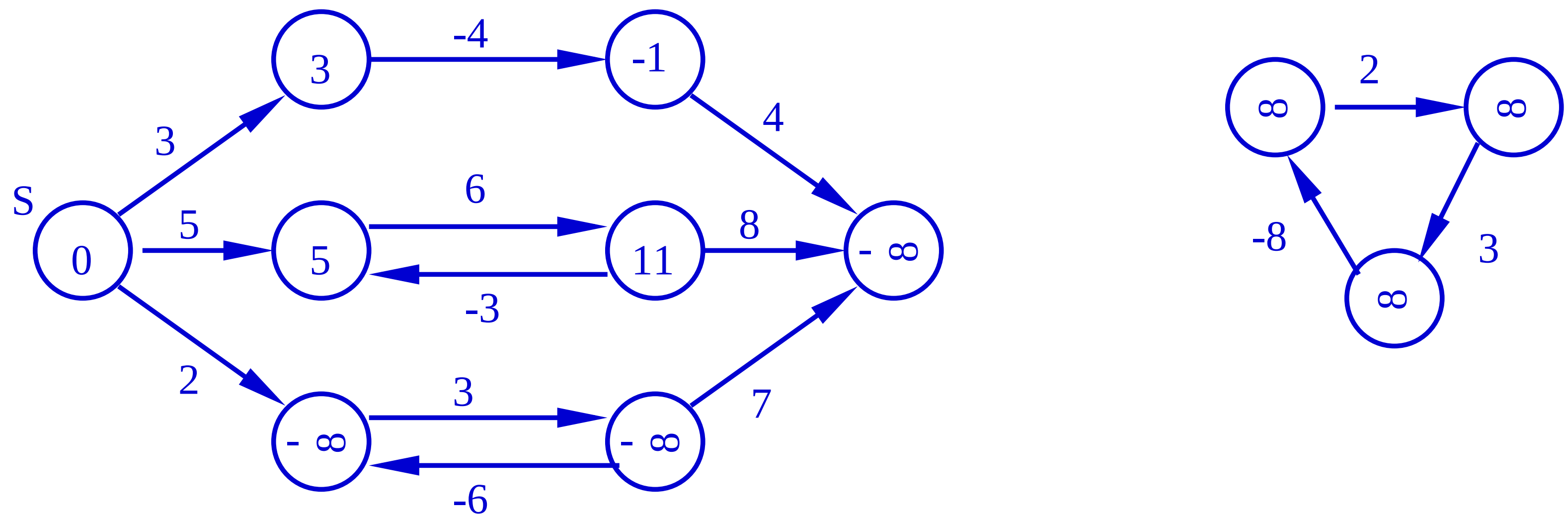
$$w(p) = \sum_{i=1}^{k} w(v_{i-1}, v_i).$$

- We define the *shortest-path weight* from u to v by

$$\delta(u, v) = \begin{cases} \min\{w(p) : p \text{ is a path from } u \text{ to } v\} \\ \infty \text{ if there is no path from } u \text{ to } v \end{cases}$$

- The *shortest path* from u to v is defined as any path p from u to v with weight $w(p) = \delta(u, v)$.
- *The problem:* Given the directed graph $G = (V, E)$ and a vertex s, find the shortest paths from s to all other vertices.
- For undirected graphs, change edge $\{u, v\}$ with weight w to a pair of edges (u, v) and (v, u) both with weight w.

Example:



- Negative weight cycle

  In some instances of the single-source shortest-paths problem, there may be edges with negative weights.

  † If there is no negative cycle, the shortest path weight $\delta(s, v)$ is still well defined.

  † If there is negative cycle reachable from s, then the shortest path weight from s to any vertex on the cycle is not well defined.

  † A lesser path can always be found by following the proposed "shortest path" and then traverse the negative weight cycle.

- Cycles in shortest path?

  † A shortest path cannot contain a negative cycle.
  Shortest path weight is not well defined.

  † A shortest path cannot contain a positive cycle.
  Removing the positive cycle will produce a path with lesser weight.

  † How about 0-weight cycle?
  We can remove all 0-weight cycles and produce a shortest path without cycle.

- We can assume that shortest paths we are looking for contain no cycle.

  Therefore any shortest path contains at most $|V| - 1$ edges.

For each vertex v, we maintain two attributes, $\pi[v]$ and $d[v]$.

- $d[v]$ is an upper bound on the weight of a shortest path from source s to v.
  - † During the execution of a shortest-path algorithm, $d[v]$ may be larger than the shortest-path weight.
  - † At the termination of a shortest-path algorithm, $d[v]$ is the shortest-path weight from s to v.
- $\pi[v]$ is used to represent the shortest paths.
  - † During the execution of a shortest-path algorithm, $\pi[]$ need not indicate shortest paths.
  - † $\pi[v]$ is the last edge of a path from s to v during the execution of a shortest-path algorithm.
  - † At the termination of a shortest-path algorithm, $\pi[v]$ represent the last edge of a shortest path from s to v.
  - † Since sub-path of a shortest path is itself shortest path, therefore $< v, \pi[v], \pi[\pi[v]], \cdots, s >$ is the shortest path from s to v in reverse order.

- Initialization

Initialize_Single_Source(G, s)

For each vertex $v \in V[G]$ do
    $d[v] = \infty$;
    $\pi[v] = nil$;
$d[s] = 0$;

- Relaxation

Relax(u, v, w)

if $d[v] > d[u] + w(u, v)$ then
    $d[v] = d[u] + w(u, v)$;
    $\pi[v] = u$;

Relax(u, v, w) tests if we can improve the shortest path to v found so far by going through u.

If so, we update $d[v]$ and $\pi[v]$.

- Each algorithm for single-source shortest-path will begin by calling Initialize_Single_Source(G, s).
- And then Relax(u, v, w) will be repeatedly applied to edges.
- The algorithms differ in how many times they relax each edge and the order in which they relax edges.

The Bellman-Ford Algorithm

Bellman-Ford algorithm solves the single-source shortest-path problem in general case where graph may contains cycles and edge weights may be negative.

- If there is no negative cycle, the algorithm will compute the shortest-paths and their weights.
- If there is negative cycle, the algorithm will report no solution exists.
- The idea is to repeatedly use the following procedure to progressively decrease an estimate $d[v]$ of the weight of shortest path from s to v.

```
Relax_All(G, s)
1    For each edge (u, v) ∈ E do
2        Relax(u, v, w);
```

Lemma: Let $p = < s = v_0, v_1, \cdots, v_k = v >$ be a path from s to v of length k and weight $w(p)$, then after k applications of Relax_All(G, s), $d[v] \leq w(p)$.

Proof:
Prove by induction on k.

- $k = 1$.
  In this case, $p = < s, v >$ and $w(p) = w(s, v)$. After Relax(s, v, w) is applied, $d[v] \leq d[s] + w(s, v) = w(s, v) = w(p)$.
- $k > 1$.

  † Let $p_1 = < v_0, v_1, \cdots, v_{k-1} >$, then p_1 is a path of length $k - 1$.

  † Therefore after $k - 1$ applications of Relax_All(G, s), we have $d[v_{k-1}] \leq w(p_1)$.

  † After another application of Relax_All(G, s),
  $d[v] \leq d[v_{k-1}] + w(v_{k-1}, v_k) \leq w(p_1) + w(v_{k-1}, v_k) = w(p)$.

□

Since shortest paths have lengths less than $|V|$, what we need to do is to apply Relax_All(G, s) $|V| - 1$ times.

```
Bellman_Ford(G, w, s)
    Initialize_SingleSource(G, s)
    for i := 1 to |V| - 1 do
        for each edge (u, v) ∈ E do
            Relax(u, v, w);
    for each edge (u, v) ∈ E do
        if d[v] > d[u] + w(u, v) then
            return False;
    return True;
```

Lines 5-7 test if the graph contains negative cycle reachable from s.

- If there is no such cycle, then there is no edge $(u, v) \in E$ such that $d[v] > d[u] + w(u, v)$ since otherwise $d[v]$ is not the shortest-path weight from s to v.
- If there is such a cycle $c =< v_0, v_1, \cdots, v_k >$ where $v_0 = v_k$ and $\sum_{i=1}^{k} w(v_{i-1}, v_i) < 0$.
  - † Suppose that (for the purpose of contradiction) for each edge $(u, v) \in E$, $d[v] \leq d[u] + w(u, v)$.
  - † Then $d[v_i] \leq d[v_{i-1}] + w(v_{i-1}, v_i)$ for $1 \leq i \leq k$.
  - † And $\sum_{i=1}^{k} d[v_i] \leq \sum_{i=1}^{k} d[v_{i-1}] + \sum_{i=1}^{k} w(v_{i-1}, v_i)$.
  - † Therefore $\sum_{i=1}^{k} w(v_{i-1}, v_i) \geq 0$
- Time complexity: $O(|V||E|)$.

Acyclic Graph

– Suppose that graph G has no cycle.

– We first use topological sorting to order the vertices of G.

- If s has label k, then for any vertex v with label $< k$, there is NO PATH from s to v, so $d[v] = \infty$.
- We then consider each vertex with label $> k$ in the order of $k+1, k+2, \cdots, |V|$
- Consider a vertex v in the above order (with label $> k$).
  We want to compute $d[v]$ and $\pi[v]$.
  We need only consider those vertices u such that (u, v) is an edge in G.

  For each $(u, v) \in E[G]$ do
      Relax(u, v, w)

- This is correct since for any $(u, v) \in E$, label for u is less the label for v.
- *Complexity:* $O(|V| + |E|)$

Non-Negative Weights

- General graph with no negative weight edge.
- Graph now is not acyclic. Therefore there is no topological order.
- What is the main idea from acyclic case?

  When we consider shortest path from s to v, the topological order enables us to ignore all vertices after v.
- Could we define an order for general graphs to do similar things?
- For general graphs,

  Order the vertices by the weights of their shortest paths from s.

  Unlike topological order, we do not know this order before we find shortest paths.

- We will find the order during the process of finding shortest paths.
- Can we first find the closest vertex w_1?

  Yes! w_1 is the vertex satisfying following:

  $w(s, w_1) = \min_v w(s, v)$

  Why?

  Consider the shortest path from s to w_1.

  It must consist of only two vertices s and w_1.

  <u>Otherwise if</u>

  $$s \to v_1 \to v_2 \to \cdots \to v_k \to w_1$$

  <u>is the shortest path from s to w_1, then $d[v_1] = w(s, v_1) \leq \delta(s, w_1) = d[w_1]$</u>

  – either w_1 is not closest – contradiction!

  – or $\delta(s, w_1) = \delta(s, v_1)$, we can choose v_1 to be the closest vertex.

  – therefore we can determine $d[w_1]$ and find w_1 this way.

- Can we find the second closest vertex w_2?

  YES! The only paths we need to consider are the edges from s (except (s, w_1)) and paths of two edges, the first one being (s, w_1), and the second one being from w_1.

  – Why? Again, consider a shortest path from s to w_2

  $$s \longrightarrow v_1 \longrightarrow v_2 \longrightarrow \cdots \longrightarrow v_k \longrightarrow w_2$$

  – Consider the first vertex (from s to w_2) that is not s and w_1.

  – It is either v_1 or v_2 (and in this case $v_1 = w_1$).

  – Therefore we choose the minimum of
  $w(s, v)$ $(v \neq w_1)$ or $d[w_1] + w(w_1, v)$ $(v \neq s)$.

  – this give us w_2 and $d[w_2]$.

Induction

Induction hypothesis:
Give graph G and a vertex s, we know the $k-1$ vertices that are closest to s and we know the weights of the shortest paths to them.

Base case: done!

Inductive Step: We want to find the kth (w_k) closest vertex and the weight of shortest path to it.
Let the $k-1$ closest vertices be $w_1, w_2, \ldots, w_{k-1}$.
Let $V_{k-1} = \{s, w_1, w_2, \ldots, w_{k-1}\}$
The shortest path from s to w_k can go only through vertices in V_{k-1}.
(If it goes through a vertex not in V_{k-1}, this vertex is closer than w_k)

Therefore w_k is the vertex satisfying the following:

$w_k \notin V_{k-1}$ and the shortest path from s to w_k through V_{k-1} is less or equal to the shortest path from s to any other vertex $v \notin V_{k-1}$ through V_{k-1}.

For $v \notin V_{k-1}$, let

$$d[v] = \min_{u \in V_{k-1}} (d[u] + w(u, v)).$$

$d[v]$ is the shortest path from s to v through V_{k-1}.

Therefore w_k is a vertex such that

$$w_k \notin V_{k-1} \text{ and } d[w_k] = \min_{v \notin V_{k-1}} \{d[v]\}.$$

- Adding w_k does not change the weights of the shortest paths from s to u, $u \in V_{k-1}$, since u is closer than w_k
- The Algorithm is complete now.

  We should consider how to implement it efficiently.

  The main computation is for $d[v]$ for $v \notin V_{k-1}$.

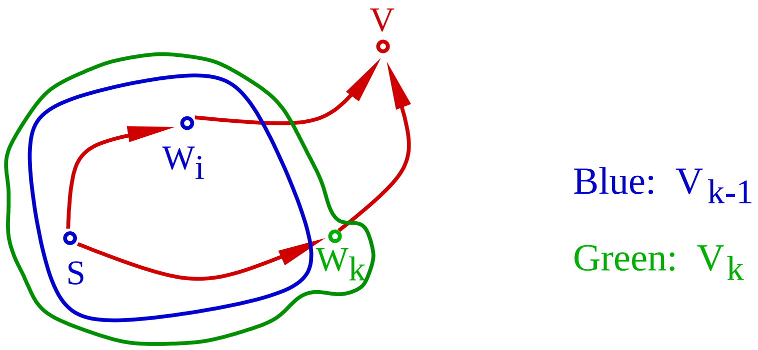
• We do not have to compute all $d[v]$ for each V_k.

Most of $d[v]$ for V_k are equal to $d[v]$ for V_{k-1}.
We only need to update a few $d[v]$ when we add w_k.

• When we add w_k
For v, such that $v \notin V_k$ and (w_k, v) is an edge.
$d[v] = \min\{d[v], \;\; d[w_k] + w(w_k, v)\}$

(Note: this is the same as Relax(w_k, v, w).)

Consider a shortest path from s to v through V_k.
If the last edge is (w_i, v), $i < k$, then there is no change to $d[v]$.
If the last edge is (w_k, v) then $d[v] = d[w_k] + w(w_k, v)$.



What data structure should we use?

Heap is a good choice!

- We can keep $d[v]$ in a min_heap. Then we can find w_k in $O(1)$ time.
- After we find w_k, we update $d[v]$.
  - Delete w_k from heap.
  - For each v in the heap such that (w_k, v) is an edge, change its key from $d[v]$ to $\min\{d[v], d[w_k] + w(w_k, v)\}$ (Relax(w_k, v, w)).
- We need to use the heap with element locations (see notes for heap)!

Dijkstra's Algorithm

The above analysis gives us the Dijkstra's algorithm.

```
Dijkstra(G, w, s)
    Initialize_Single_Source(G, s);
    S := ∅;
    Q := V[G];
    while Q ≠ ∅ do
        u := Extract_Min(Q);
        S := S ∪ {u};
        for each (u, v) ∈ E do
            Relax(u, v, w);
            Update v in Q;
```

$\boxed{\text{Time Complexity}}$

With a binary heap:
$|V|$ delete min operations: $O(|V| \log(|V|))$
$|E|$ update operations: $O(|E| \log(|V|))$
TOTAL $O((|V| + |E|) \log(|V|))$

With a Fibonacci heap:
$|V|$ delete min operations: $O(|V| \log(|V|))$
$|E|$ update operations: $O(|E|)$
TOTAL $O(|V| \log(|V|) + |E|)$

Without a heap:
$|V|$ delete min operations: $O(|V||V|)$
$|E|$ update operations: $O(|E|)$
TOTAL $O(|V|^2 + |E|) = O(|V|^2)$

(Compare with acyclic case $O(|V| + |E|)$)
(Compare with Bellman-Ford algorithm $O(|V||E|)$)

Minimum Spanning Trees

- Consider an undirected weighted graph $G = (V, E)$.
- A spanning tree of G is a connected subgraph that contains all vertices and no cycles.
- Minimum spanning tree of G: a spanning tree T of G such that the sum of the weights of edges in T is minimum.
- Applications:
  - *computer networks (e.g. broadcast path)*
  - there is a cost for sending a message on the link.
  - broadcast a message to all computers in the network from an arbitrary computer
  - want to minimize the cost

The Problem

Given an undirected connected weighted graph $G = (V, E)$, find a spanning tree T of G of minimum cost.

Idea.
Extend tree: always choose to extend tree by adding cheapest edge.

For simplicity, we assume all costs (weights) are distinct!

Base case: Let r be an arbitrarily chosen root vertex. The minimum-cost edge incident to r must be in the minimum spanning tree (MST)

† Suppose this edge is $\{r, s\}$

† if $\{r, s\}$ is not in MST, add $\{r, s\}$ to MST

† Now we have a cycle

† Delete the MST edge incident to r from the cycle. We have a new tree.

† the cost of this new tree is less than the cost of MST. Contradiction!

Induction hypothesis

Given a connected graph $G = (V, E)$, we know how to find a subgraph T of G with k edges, such that T is a tree and T is a subgraph of the MST of G.

Extend T:

† Find the cheapest edge from a vertex in T to a vertex not in T. Let it be $\{u, v\}$, such that $u \in T$ and $v \notin T$.

† Add $\{u, v\}$ to T.

† Claim: We now have a tree with $k + 1$ edges which is a subgraph of the MST of G.

- Again add $\{u, v\}$ to the MST
- Consider the path from u to v in MST
- There must be an edge $e = \{u_1, v_1\}$ in this path such that $u_1 \in T$ and $v_1 \notin T$.
- Delete edge e
- Since weight$(e) >$ weight$(\{u, v\})$, the new tree has a cost less than the MST
- Contradiction

Implementation

- Similar to the implementation of single-source shortest-path algorithm
- Choose an arbitrary vetex as the root
- For each iteration we need to find the minimum cost edge connecting T to vertices outside of T.
- We again use a heap.

  For each vertex w not in T, we use the minimum-cost of the costs of the edges going into w from a vertex in T as the key.
- For each iteration we delete min from the heap. Suppose u is the new vertex.

  Update the keys for vertex v not in T by cost of edge $\{u, v\}$.
- Time: $|V|$ delete min: $O(|V| \log(|V|))$

  $|E|$ update operations: $O(|E| \log(|V|))$

  Total: $O((|V| + |E|) \log(|V|))$
- This is called PRIMS algorithm

Prim's Algorithm

The above analysis gives us the Prim's algorithm.

MST_Prim(G, w, r)
for each $u \in V[G]$ do
    $key[u] := \infty$;
    $\pi[u] :=$ NIL;
$key[r] := 0$;
$Q := V[G]$;
while $Q \neq \emptyset$ do
    $u :=$ Extract_Min(Q);
    for each $v \in Adj[u]$ do
        if $v \in Q$ and $w(u, v) < key[v]$ then
            $\pi[v] := u$;
            $key[v] := w(u, v)$;
            update $key[v]$ in Q

$\boxed{\text{Kruskal's MST}}$

Idea: Choose cheapest edge in a graph.

Algorithm:

put all edges in a heap, put each vertex in a set by itself;
while not found a MST yet **do begin**
  delete min edge, $\{u, v\}$, from the heap;
  if u and v are not in the same set
    mark $\{u, v\}$ as tree edge;
    union sets containing u and v;
  if u and v are in the same set
    do nothing;
end

<u>Time:</u>
$O((|V| + |E|)\log(|V|))$ for heap operation.
$O(|E|\log^*(|V|)$ for union-find operation.
Total: $O((|V| + |E|)\log(|V|))$ time.

All-Pair Shortest-Paths Problem

- *The problem:* Given a weighted graph $G = (V, E)$, find the shortest paths between all pairs of vertices.
- We can call single-source shortest-paths algorithm $|V|$ times

  † If there is no negative cycle.
  Complexity: $O(|V|^2|E|)$

  † If there is no negative weight edge.
  Complexity: $O(|V|^2 \log(|V|) + |V||E|)$ or $O(|V|(|V| + |E|) \log(|V|))$
  If G is not dense, this is a good solution.

- We consider to use induction to design a direct solution.

- We can use induction on the vertices.
- We know the shortest paths between a set of k vertices (V_k).
- We want to add a new vertex u
- We can find the shortest path from u to all the vertices in V_k

shortest-path$(u, w) =$
$\min_{v \in V_K, (u,v) \in E} \{w(u, v) + \text{shortest-path}(v, w)\} (*)$

Shortest-path(w, u) can be computed similarly!

We update shortest-path(w_1, w_2), $w_1, w_2 \in V_k$

shortest-path$(w_1, w_2) = \min\{$shortest-path$(w_1, u)+$ shortest-path(u, w_2),
shortest-path$(w_1, w_2)\}$ (**)

Time: (**) can be done in $|V|^2$
(*) can be done in $|V|^2$

Total: $O(|V|^3)$.

A better solution

– *Idea:* Number of vertices is fixed.

  Induction puts restrictions on the type of paths allowed

– We label vertices from 1 to $|V|$

  A path from u to w is called a k-path if, except for u and w, the highest-labelled vertex on the path is labelled by k.

  A 0-path is an edge

– *Induction hypothesis:*

  We know the lengths of the shortest paths between all pairs of vertices such that only k-paths, for some $k \leq m$ are considered.

– *Base case:* $m = 0$

  only direct edges can be considered

$\boxed{\text{Inductive step}}$

(extend $m - 1$ to m)

We consider all k-paths such that $k \leq m$.
The only new paths are m-paths.
Let the vertex with label m be v_m.
Consider a shortest m-path between u and v.

<u>This m-path must include v_m only once!</u>

Therefore this m-path is a shortest k-path (for some $k \leq m - 1$) between u and v_m appended by a shortest j-path (for some $j \leq m - 1$) from v_m to v.
By induction we already know the length of the k-path and the j-path!
We update shortest-path (u, v) by:

min{shortest-path(u, v_m) + shortest-path(v_m, v), shortest-path(u, v)}

This leads to a very simple program! (Floyd-Warshall algorithm)

```
for x := 1 to |V| do   { base case }
    for y := 1 to |V| do
        if (x, y) ∈ E, then
            d[x, y] := w(x, y);
        else
            d[x, y] := ∞;

for x := 1 to |V| do
    d[x, x] := 0;

for m := 1 to |V| do   { the induction sequence }
    for x := 1 to |V| do
        for y := 1 to |V| do
            if d[x, m] + d[m, y] < d[x, y] then
                d[x, y] := d[x, m] + d[m, y]
```

Time: $O(|V|^3)$. Again, if the graph is sparse, then $O(|V|^2 \log(|V|) + |V||E|)$ is a better solution when there is no negative weight.

If we need to find the shortest paths not just the weights. Let $\phi[i,j]$ be highest numbered vertex on the shortest path from i to j.

for $x := 1$ to $|V|$ do { base case }
  for $y := 1$ to $|V|$ do
    if $(x, y) \in E$, then
      $d[x, y] := w(x, y)$; $\phi[x, y] := x$;
    else
      $d[x, y] := \infty$; $\phi[x, y] :=$Nil;

for $x := 1$ to $|V|$ do
  $d[x, x] := 0$; $\phi[x, x] :=$Nil;

for $m := 1$ to $|V|$ do { the induction sequence }
  for $x := 1$ to $|V|$ do
    for $y := 1$ to $|V|$ do
      if $d[x, m] + d[m, y] < d[x, y]$ then
        $d[x, y] := d[x, m] + d[m, y]$;
        $\phi[x, y] := m$;

Time: $O(|V|^3)$

If we need to find the shortest paths not just the weights. Let $\pi[i,j]$ be the predecessor of j on the shortest path from i to j.

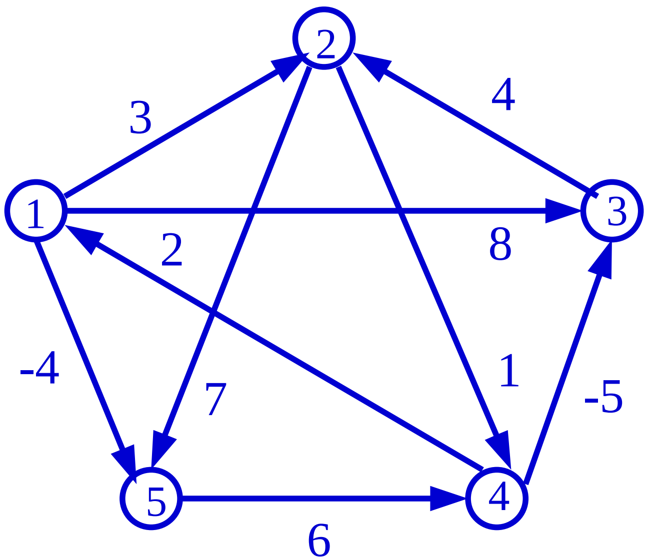
for $x := 1$ to $|V|$ do { base case }
    for $y := 1$ to $|V|$ do
        if $(x,y) \in E$, then
            $d[x,y] := w(x,y);\quad \pi[x,y] := x;$
        else
            $d[x,y] := \infty;\quad \pi[x,y] :=$Nil;

for $x := 1$ to $|V|$ do
    $d[x,x] := 0;\quad \pi[x,x] :=$Nil;

for $m := 1$ to $|V|$ do { the induction sequence }
    for $x := 1$ to $|V|$ do
        for $y := 1$ to $|V|$ do
            if $d[x,m] + d[m,y] < d[x,y]$ then
                $d[x,y] := d[x,m] + d[m,y];$
                $\pi[x,y] := \pi[m,y];$

Time: $O(|V|^3)$

Example: Figure 25.1.



$$D^{(0)} = \begin{pmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix} \qquad \Phi^{(0)} = \begin{pmatrix} \mathrm{NIL} & 1 & 1 & \mathrm{NIL} & 1 \\ \mathrm{NIL} & \mathrm{NIL} & \mathrm{NIL} & 2 & 2 \\ \mathrm{NIL} & 3 & \mathrm{NIL} & \mathrm{NIL} & \mathrm{NIL} \\ 4 & \mathrm{NIL} & 4 & \mathrm{NIL} & \mathrm{NIL} \\ \mathrm{NIL} & \mathrm{NIL} & \mathrm{NIL} & 5 & \mathrm{NIL} \end{pmatrix}$$

$$D^{(1)} = \begin{pmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix} \qquad \Phi^{(1)} = \begin{pmatrix} \mathrm{NIL} & 1 & 1 & \mathrm{NIL} & 1 \\ \mathrm{NIL} & \mathrm{NIL} & \mathrm{NIL} & 2 & 2 \\ \mathrm{NIL} & 3 & \mathrm{NIL} & \mathrm{NIL} & \mathrm{NIL} \\ 4 & 1 & 4 & \mathrm{NIL} & 1 \\ \mathrm{NIL} & \mathrm{NIL} & \mathrm{NIL} & 5 & \mathrm{NIL} \end{pmatrix}$$

$$D^{(2)} = \begin{pmatrix} 0 & 3 & 8 & 4 & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix} \qquad \Phi^{(2)} = \begin{pmatrix} \mathrm{NIL} & 1 & 1 & 2 & 1 \\ \mathrm{NIL} & \mathrm{NIL} & \mathrm{NIL} & 2 & 2 \\ \mathrm{NIL} & 3 & \mathrm{NIL} & 2 & 2 \\ 4 & 1 & 4 & \mathrm{NIL} & 1 \\ \mathrm{NIL} & \mathrm{NIL} & \mathrm{NIL} & 5 & \mathrm{NIL} \end{pmatrix}$$

$$D^{(3)} = \begin{pmatrix} 0 & 3 & 8 & 4 & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & -1 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix} \qquad \Phi^{(3)} = \begin{pmatrix} \mathrm{NIL} & 1 & 1 & 2 & 1 \\ \mathrm{NIL} & \mathrm{NIL} & \mathrm{NIL} & 2 & 2 \\ \mathrm{NIL} & 3 & \mathrm{NIL} & 2 & 2 \\ 4 & 3 & 4 & \mathrm{NIL} & 1 \\ \mathrm{NIL} & \mathrm{NIL} & \mathrm{NIL} & 5 & \mathrm{NIL} \end{pmatrix}$$

$$D^{(4)} = \begin{pmatrix} 0 & 3 & -1 & 4 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 3 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{pmatrix} \qquad \Phi^{(4)} = \begin{pmatrix} \mathrm{NIL} & 1 & 4 & 2 & 1 \\ 4 & \mathrm{NIL} & 4 & 2 & 4 \\ 4 & 3 & \mathrm{NIL} & 2 & 4 \\ 4 & 3 & 4 & \mathrm{NIL} & 1 \\ 4 & 4 & 4 & 5 & \mathrm{NIL} \end{pmatrix}$$

$$D^{(5)} = \begin{pmatrix} 0 & 1 & -3 & 2 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 3 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{pmatrix} \qquad \Phi^{(5)} = \begin{pmatrix} \mathrm{NIL} & 5 & 5 & 5 & 1 \\ 4 & \mathrm{NIL} & 4 & 2 & 1 \\ 4 & 3 & \mathrm{NIL} & 2 & 1 \\ 4 & 3 & 4 & \mathrm{NIL} & 1 \\ 4 & 3 & 4 & 5 & \mathrm{NIL} \end{pmatrix}$$

$$D^{(0)} = \begin{pmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix} \qquad \Pi^{(0)} = \begin{pmatrix} \mathrm{NIL} & 1 & 1 & \mathrm{NIL} & 1 \\ \mathrm{NIL} & \mathrm{NIL} & \mathrm{NIL} & 2 & 2 \\ \mathrm{NIL} & 3 & \mathrm{NIL} & \mathrm{NIL} & \mathrm{NIL} \\ 4 & \mathrm{NIL} & 4 & \mathrm{NIL} & \mathrm{NIL} \\ \mathrm{NIL} & \mathrm{NIL} & \mathrm{NIL} & 5 & \mathrm{NIL} \end{pmatrix}$$

$$D^{(1)} = \begin{pmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix} \qquad \Pi^{(1)} = \begin{pmatrix} \mathrm{NIL} & 1 & 1 & \mathrm{NIL} & 1 \\ \mathrm{NIL} & \mathrm{NIL} & \mathrm{NIL} & 2 & 2 \\ \mathrm{NIL} & 3 & \mathrm{NIL} & \mathrm{NIL} & \mathrm{NIL} \\ 4 & 1 & 4 & \mathrm{NIL} & 1 \\ \mathrm{NIL} & \mathrm{NIL} & \mathrm{NIL} & 5 & \mathrm{NIL} \end{pmatrix}$$

$$D^{(2)} = \begin{pmatrix} 0 & 3 & 8 & 4 & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix} \qquad \Pi^{(2)} = \begin{pmatrix} \mathrm{NIL} & 1 & 1 & 2 & 1 \\ \mathrm{NIL} & \mathrm{NIL} & \mathrm{NIL} & 2 & 2 \\ \mathrm{NIL} & 3 & \mathrm{NIL} & 2 & 2 \\ 4 & 1 & 4 & \mathrm{NIL} & 1 \\ \mathrm{NIL} & \mathrm{NIL} & \mathrm{NIL} & 5 & \mathrm{NIL} \end{pmatrix}$$

$$D^{(3)} = \begin{pmatrix} 0 & 3 & 8 & 4 & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & -1 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix} \qquad \Pi^{(3)} = \begin{pmatrix} \text{NIL} & 1 & 1 & 2 & 1 \\ \text{NIL} & \text{NIL} & \text{NIL} & 2 & 2 \\ \text{NIL} & 3 & \text{NIL} & 2 & 2 \\ 4 & 3 & 4 & \text{NIL} & 1 \\ \text{NIL} & \text{NIL} & \text{NIL} & 5 & \text{NIL} \end{pmatrix}$$

$$D^{(4)} = \begin{pmatrix} 0 & 3 & -1 & 4 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 3 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{pmatrix} \qquad \Pi^{(4)} = \begin{pmatrix} \text{NIL} & 1 & 4 & 2 & 1 \\ 4 & \text{NIL} & 4 & 2 & 1 \\ 4 & 3 & \text{NIL} & 2 & 1 \\ 4 & 3 & 4 & \text{NIL} & 1 \\ 4 & 3 & 4 & 5 & \text{NIL} \end{pmatrix}$$

$$D^{(5)} = \begin{pmatrix} 0 & 1 & -3 & 2 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 3 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{pmatrix} \qquad \Pi^{(5)} = \begin{pmatrix} \text{NIL} & 3 & 4 & 5 & 1 \\ 4 & \text{NIL} & 4 & 2 & 1 \\ 4 & 3 & \text{NIL} & 2 & 1 \\ 4 & 3 & 4 & \text{NIL} & 1 \\ 4 & 3 & 4 & 5 & \text{NIL} \end{pmatrix}$$

If we need to find the shortest paths and shortest cycles, let $\pi[i, j]$ be the predecessor of j on the shortest path from i to j.

for $x := 1$ to $|V|$ do { base case }
    for $y := 1$ to $|V|$ do
        if $(x, y) \in E$, then
            $d[x, y] := w(x, y)$; $\pi[x, y] := x$;
        else
            $d[x, y] := \infty$; $\pi[x, y] :=$Nil;

for $m := 1$ to $|V|$ do { the induction sequence }
    for $x := 1$ to $|V|$ do
        for $y := 1$ to $|V|$ do
            if $d[x, m] + d[m, y] < d[x, y]$ then
                $d[x, y] := d[x, m] + d[m, y]$;
                $\pi[x, y] := \pi[m, y]$;

Time: $O(|V|^3)$

$$D^{(0)} = \begin{pmatrix} \infty & 3 & 8 & \infty & -4 \\ \infty & \infty & \infty & 1 & 7 \\ \infty & 4 & \infty & \infty & \infty \\ 2 & \infty & -5 & \infty & \infty \\ \infty & \infty & \infty & 6 & \infty \end{pmatrix} \qquad \Pi^{(0)} = \begin{pmatrix} \mathrm{NIL} & 1 & 1 & \mathrm{NIL} & 1 \\ \mathrm{NIL} & \mathrm{NIL} & \mathrm{NIL} & 2 & 2 \\ \mathrm{NIL} & 3 & \mathrm{NIL} & \mathrm{NIL} & \mathrm{NIL} \\ 4 & \mathrm{NIL} & 4 & \mathrm{NIL} & \mathrm{NIL} \\ \mathrm{NIL} & \mathrm{NIL} & \mathrm{NIL} & 5 & \mathrm{NIL} \end{pmatrix}$$

$$D^{(1)} = \begin{pmatrix} \infty & 3 & 8 & \infty & -4 \\ \infty & \infty & \infty & 1 & 7 \\ \infty & 4 & \infty & \infty & \infty \\ 2 & 5 & -5 & \infty & -2 \\ \infty & \infty & \infty & 6 & \infty \end{pmatrix} \qquad \Pi^{(1)} = \begin{pmatrix} \mathrm{NIL} & 1 & 1 & \mathrm{NIL} & 1 \\ \mathrm{NIL} & \mathrm{NIL} & \mathrm{NIL} & 2 & 2 \\ \mathrm{NIL} & 3 & \mathrm{NIL} & \mathrm{NIL} & \mathrm{NIL} \\ 4 & 1 & 4 & \mathrm{NIL} & 1 \\ \mathrm{NIL} & \mathrm{NIL} & \mathrm{NIL} & 5 & \mathrm{NIL} \end{pmatrix}$$

$$D^{(2)} = \begin{pmatrix} \infty & 3 & 8 & 4 & -4 \\ \infty & \infty & \infty & 1 & 7 \\ \infty & 4 & \infty & 5 & 11 \\ 2 & 5 & -5 & 6 & -2 \\ \infty & \infty & \infty & 6 & \infty \end{pmatrix} \qquad \Pi^{(2)} = \begin{pmatrix} \mathrm{NIL} & 1 & 1 & 2 & 1 \\ \mathrm{NIL} & \mathrm{NIL} & \mathrm{NIL} & 2 & 2 \\ \mathrm{NIL} & 3 & \mathrm{NIL} & 2 & 2 \\ 4 & 1 & 4 & 2 & 1 \\ \mathrm{NIL} & \mathrm{NIL} & \mathrm{NIL} & 5 & \mathrm{NIL} \end{pmatrix}$$

$$D^{(3)} = \begin{pmatrix} \infty & 3 & 8 & 4 & -4 \\ \infty & \infty & \infty & 1 & 7 \\ \infty & 4 & \infty & 5 & 11 \\ 2 & -1 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & \infty \end{pmatrix} \qquad \Pi^{(3)} = \begin{pmatrix} \text{NIL} & 1 & 1 & 2 & 1 \\ \text{NIL} & \text{NIL} & \text{NIL} & 2 & 2 \\ \text{NIL} & 3 & \text{NIL} & 2 & 2 \\ 4 & 3 & 4 & 2 & 1 \\ \text{NIL} & \text{NIL} & \text{NIL} & 5 & \text{NIL} \end{pmatrix}$$

$$D^{(4)} = \begin{pmatrix} 6 & 3 & -1 & 4 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 3 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 4 \end{pmatrix} \qquad \Pi^{(4)} = \begin{pmatrix} 4 & 1 & 4 & 2 & 1 \\ 4 & 3 & 4 & 2 & 1 \\ 4 & 3 & 4 & 2 & 1 \\ 4 & 3 & 4 & 2 & 1 \\ 4 & 3 & 4 & 5 & 1 \end{pmatrix}$$

$$D^{(5)} = \begin{pmatrix} 4 & 1 & -3 & 2 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 3 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 4 \end{pmatrix} \qquad \Pi^{(5)} = \begin{pmatrix} 4 & 3 & 4 & 5 & 1 \\ 4 & 3 & 4 & 2 & 1 \\ 4 & 3 & 4 & 2 & 1 \\ 4 & 3 & 4 & 2 & 1 \\ 4 & 3 & 4 & 5 & 1 \end{pmatrix}$$