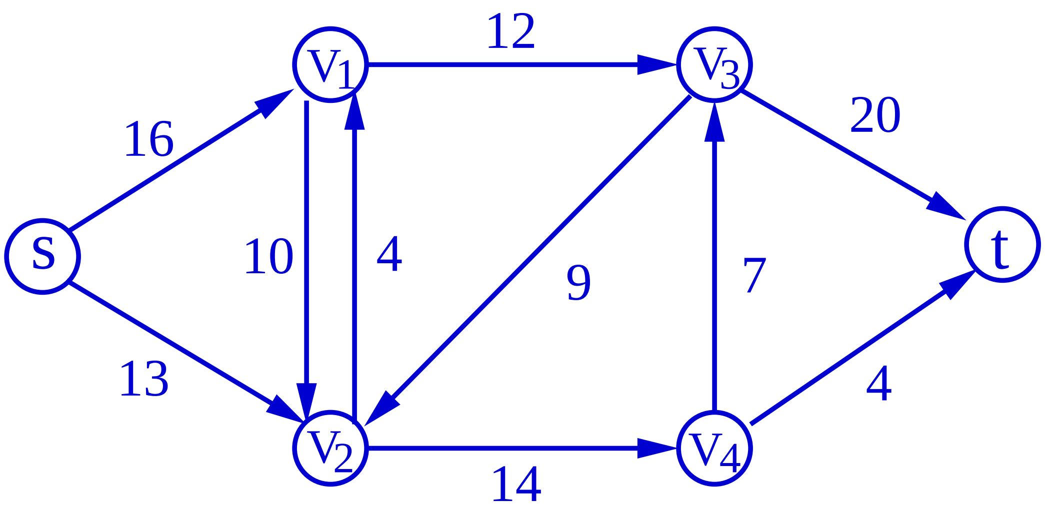
Maximum Flow

Flow Network

- The following figure shows an example of a flow network:



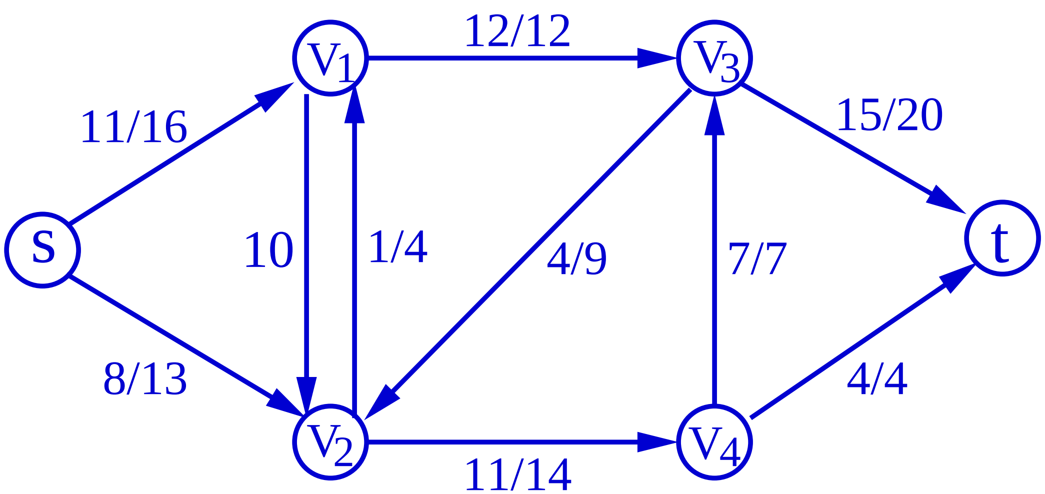
- A flow network $G = (V, E)$ is a directed graph. Each edge $(u, v) \in E$ has a nonnegative capacity $c(u, v) \geq 0$. $c(u, v)$ is possibly not equal to $c(v, u)$. By convention, we say $c(u, v) = 0$ if $(u, v) \notin E$.

- There is one **source** vertex and one **sink** vertex in a flow network. We denote them by s and t, respectively.

- We want to find a "flow" with maximum value that flows from the source to the target.
- Maximum Flow is a very practical problem.
- Many computational problems can be reduced to a Maximum Flow problem.

A Flow

- For any vertex v, we assume that there is a path from s to v and a path from v to t.
- A flow in G is a function $f : V \times V \rightarrow \mathbf{R}$ that specifies the direct flow value between every two nodes.



- f should satisfy the following three properties before it can be called as a flow.
  - **Capacity constraint:** For all $u, v \in V$, $f(u, v) \leq c(u, v)$.
  - **Skew symmetry:** For all $u, v \in V$, $f(u, v) = -f(v, u)$.
  - **Flow conservation:** For all $u \in V - \{s, t\}$, $\sum_{v \in V} f(u, v) = 0$.

If $(u, v) \notin E$ and $(v, u) \notin E$, then $c(u, v) = c(v, u) = 0$.

By capacity constraint, $f(u, v) \leq 0$ and $f(v, u) \leq 0$.

By skey symmetry, $f(u, v) \geq 0$ and $f(v, u) \geq 0$.

Therefore $f(u, v) = f(v, u) = 0$.

If there is no edge between u and v, then there is no flow between u and v.

- The value of the flow f, denoted by $|f|$, is defined by

$$|f| = \sum_{v \in V} f(s, v).$$

- $|f|$ is the total flow out of the source.
-

Lemma 1.

$$|f| = \sum_{u \in V} f(u, t).$$

That is, the flow out of the source is equal to the flow into the sink.

Proof.

(1) $\sum_{u \in V} \sum_{v \in V} f(u, v) = 0$. (Skew symmetry)

(2) $\sum_{u \in V - \{s,t\}} \sum_{v \in V} f(u, v) = 0$. (Flow conservation)

(3) $\sum_{u \in \{s,t\}} \sum_{v \in V} f(u, v) = 0$.

(4) $\sum_{v \in V} f(s, v) = -\sum_{v \in V} f(t, v) = \sum_{v \in V} f(v, t)$.

□

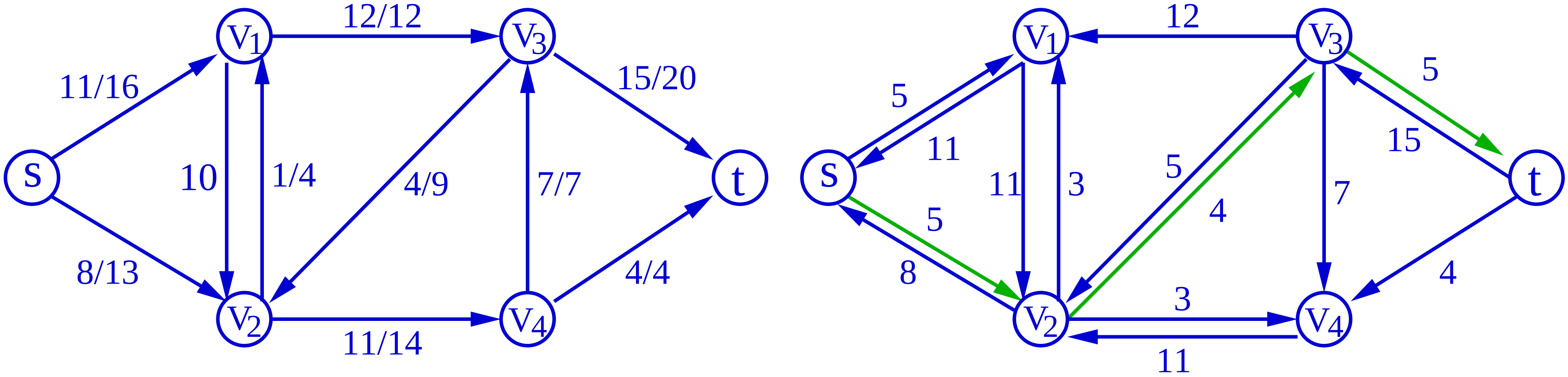
Idea of the Ford-Fulkerson method

- The Ford-Fulkerson method is the standard method for solving a maximum-flow problem.
- The idea of the method is "iterative improvement". We start with an arbitrary flow. Then we check whether an improvement is possible.
- Suppose we start with an empty flow. The improvement is a path from the source to the sink.
- What if the current flow is not empty?

Residual network

- We need to examine the "residual capacity" for each edge.
- We check whether there is a path $s \to t$ such that all edges on the path have a positive "residual capacity".
- If so, we increase the flow. If not, we have got a *maximal* solution.
- Given a flow network G. Let f be a flow. The residual capacity of (u, v) is given by $c_f(u, v) = c(u, v) - f(u, v)$.
- The residual network induced by f is $G_f = (V, E_f)$, where $E_f = \{(u, v) \in V \times V : c_f(u, v) > 0\}$.
- If there is a path from s to t in the residual network, then there is room to improve the current flow.

A flow in a flow network and its residual network.



- Note that if both (u, v) and (v, u) are not in the original flow network G, neither (u, v) nor (v, u) can appear in the residual network. Therefore, $|E_f| \leq 2|E|$.
- Let f' be a flow in the residual network G_f. We can define a new flow $(f + f')$ in G, as follows

$$(f + f')(u, v) = f(u, v) + f'(u, v).$$

-

Lemma 2. *$f + f'$ is a flow in G.*

Proof.

We need to verify the three constraints:

(1) Capacity constraint: $(f + f')(u, v) \leq c(u, v)$.

(2) Skew symmetry: $(f + f')(u, v) = -(f + f')(v, u)$.

(3) Flow conservation: For all $u \in V - \{s, t\}$, $\sum_{v \in V} (f + f')(u, v) = 0$.

□

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  Lemma 3. *The value of the new flow $f + f'$ is equal to total values of f and f'. I.e., $|f + f'| = |f| + |f'|$.*

- *Proof.*

$$
\begin{aligned}
|f + f'| &= \sum_{v \in V} (f + f')(s, v) \\
&= \sum_{v \in V} (f(s, v) + f'(s, v)) \\
&= \sum_{v \in V} f(s, v) + \sum_{v \in V} f'(s, v)) \\
&= |f| + |f'|
\end{aligned}
$$

□

Augmenting path

- Given a flow network $G = (V, E)$ and a flow f in G, an augmenting path is a simple path from s to t in the residual graph G_f.
- An augmenting path admits some additional positive flow for each edge on the path.
- The residual capacity of an augmenting path p is defined as

$$c_f(p) = \min\{c_f(u,v) : (u,v) \text{ is in } p\}$$

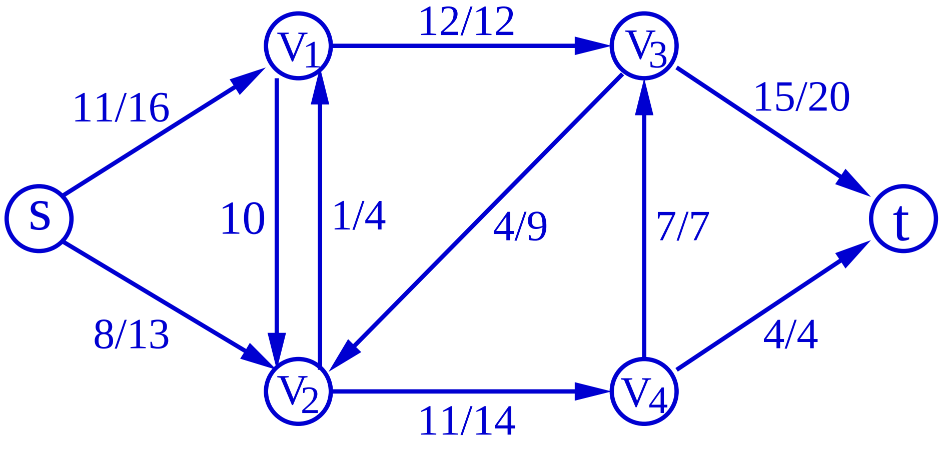
- $c_f(p)$ is the maximum amount of additional flow we can increase through path p.

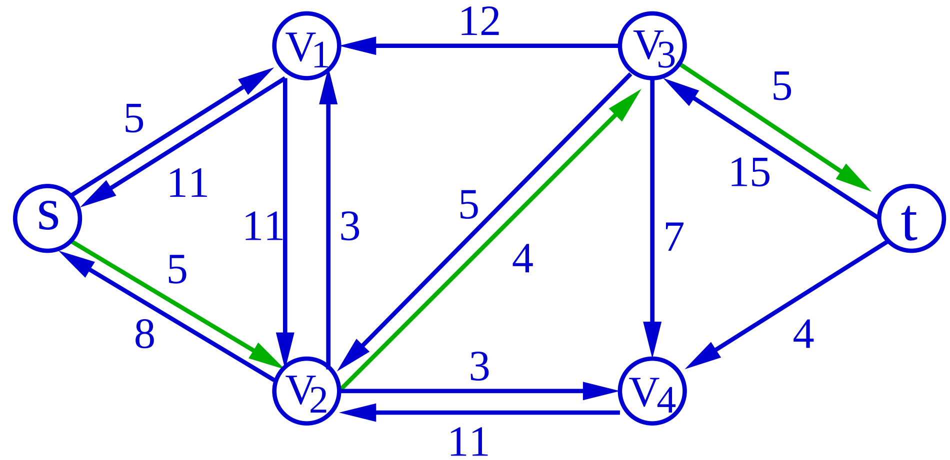
Lemma 4. *Let $G = (V, E)$ be a flow network, let f be a flow in G, and let p be an augmenting path in G_f. Define a function $f_p : V \times V \to R$ by*

$$f_p(u,v) = \begin{cases} c_f(p) & \text{if } (u,v) \text{ is on } p, \\ -c_f(p) & \text{if } (v,u) \text{ is on } p, \\ 0 & \text{otherwise.} \end{cases}$$

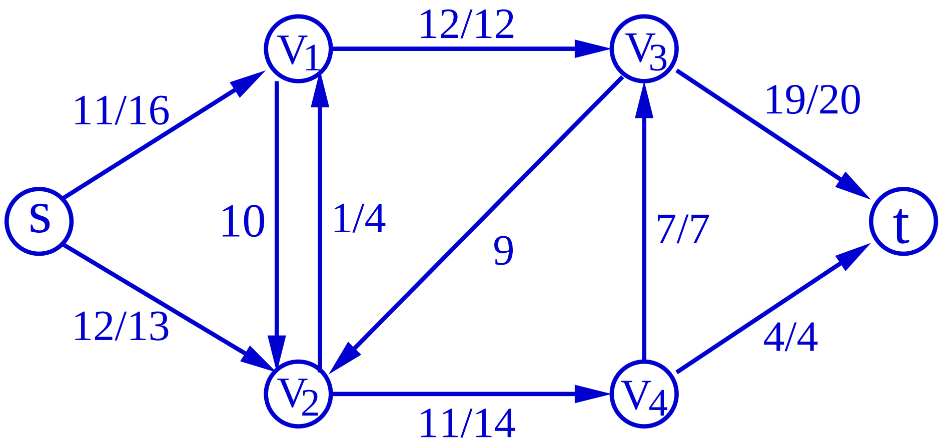
Then, f_p is a flow in G_f with value $|f_p| = c_f(p) > 0$.

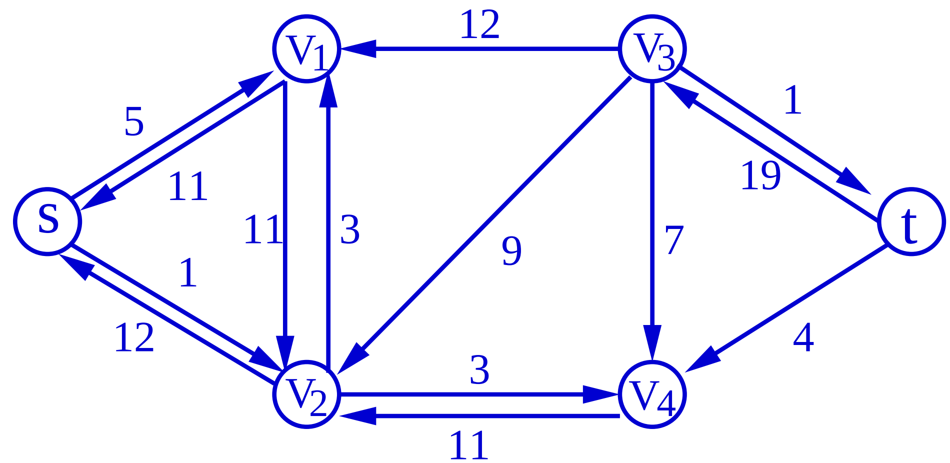
A flow in a flow network and its residual network.





A new flow from the augmenting path and its residual network.





The basic Ford-Fulkerson algorithm

- Ford-Fulkerson(G,s,t)
  1. **for** each edge $(u, v) \in E$
  2. $\quad f[u, v] \leftarrow 0$, $f[v, u] \leftarrow 0$.
  3. **while** there exists a path p from s to t in the residual network G_f
  4. $\quad c_f(p) \leftarrow \min\{c_f(u, v) : (u, v) \text{ is in } p\}$.
  5. $\quad$ **for** each edge (u, v) in p
  6. $\qquad f[u, v] \leftarrow f[u, v] + c_f(p)$
  7. $\qquad f[v, u] \leftarrow -f[u, v]$

- The path p from s to t in the residual network G_f is called the augmenting path.
- The augmenting path p defines a flow in G_f. By adding this flow f_p to the current flow f, we get a better flow $f + f_p$ with value $|f| + |f_p|$.
- Figure 26.6 on p.726-627 of the textbook shows an example.

Is the solution optimal?

- We have found an intuitive algorithm to provide a *maximal* flow. But is this flow *maximum*?
- Although we cannot increase the current flow by augmenting paths, is it possible that we find a completely different flow which has a better value?
- It turns out that the solution found by the Ford-Fulkerson algorithm is the maximum one.
- But we want to prove it.

Working with flows

- Let f be a flow. The flow from one set of vertices, X, to another set Y, is defined by $f(X, Y) = \sum_{x \in X} \sum_{y \in Y} f(x, y)$.
- **Lemma 5.** *Let $G = (V, E)$ be a flow network and let f be a flow on G, then;*
  (1) For all $X \subset V$, $f(X, X) = 0$.
  (2) For all $X, Y \subset V$, $f(X, Y) = -f(Y, X)$.
  (3) For all $X, Y, Z \subset V$ with $X \cap Y = \emptyset$, $f(X \cup Y, Z) = f(X, Z) + f(Y, Z)$ and $f(Z, X \cup Y) = f(Z, X) + f(Z, Y)$.
- *Proof.*

□

Cuts of flow networks

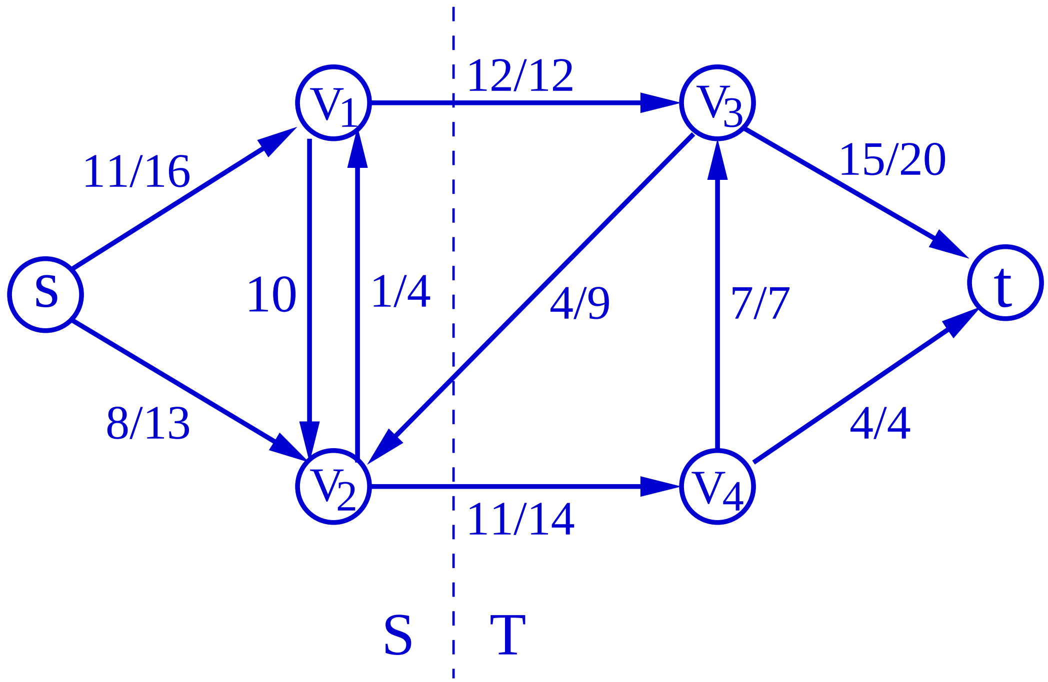
- A cut (S, T) in the flow network $G = (V, E)$ is a partition of V into S and $T = V - S$ such that $s \in S$ and $t \in T$.
- The *net flow* across the cut (S, T) is defined to be

$$f(S, T) = \sum_{u \in S} \sum_{v \in T} f(u, v).$$

- The capacity of the cut (S, T) is defined to be

$$c(S, T) = \sum_{u \in S} \sum_{v \in T} c(u, v).$$

- Obviously, $f(S, T) \leq c(S, T)$.



Lemma 6. *Let f be a flow in flow network G. Let (S, T) be any cut of G. Then the net flow across (S, T) is $f(S, T) = |f|$.*

Proof.
By flow conservation, we have $f(S - \{s\}, V) = 0$.
Also, $f(S, V) = f(S, S) + f(S, T) = f(S, T)$.
Therefore, $f(S, T) = f(S, V) = f(S - \{s\}, V) + f(\{s\}, V) = f(\{s\}, V) = |f|$.
□

- Therefore, the maximum flow is bounded by the capacity of the "minimum" cut.

Theorem 1. *If f is a flow in a flow network $G = (V, E)$ with source s and sink t, then the following conditions are equivalent:*
1. f is a maximum flow in G.
2. The residual network G_f contains no augmenting paths.

Proof. (1) $\Rightarrow$ (2): Obvious, because the existence of augmenting paths means a better flow exists.
(2) $\Rightarrow$ (1): G_f has no path from s to t. Let S be all the vertices that can be reached from s, and $T = V - S$. Then (S, T) is a cut.
For each $u \in S$ and $v \in T$, $f(u, v) = c(u, v)$. Therefore, $f(S, T) = c(S, T)$. But we know that $f^*(S, T) \leq c(S, T)$ for any flow f^*. Hence we conclude that f is the maximum. □

Exercise: Read the proof of Theorem 26.6 at p.723 of the textbook. The proof there is essentially the same but in a different form.

Corollary 1. *The Ford-Fulkerson algorithm gives the maximum flow of a flow network.*

Complexity

- Assuming that the capacities are integers.
- Every augmenting path will increase the flow by at least 1. So, the **while** loop will be repeated $O(|f^*|)$ time, where f^* is the maximum flow.
- The time complexity is $O(|E| \times |f^*|)$.
- Figure 26.7 on p.728 of textbook shows a worst case example.

Edmonds-Karp algorithm

- The Edmonds-Karp algorithm is almost the same as the Ford-Fulkerson algorithm.
- The difference is that we find the shortest path (in terms of number of edges) from s to t in the residual graph, and use the shortest path as the augmenting path.
- The worst case running time is reduced to $O(|V| \times |E|^2)$.
- Proof is omitted. See p.729 of text book if you are interested to know.

Applications

- The maximum-bipartite-matching problem.

  Example: m boys and n girls are attending a dance party. Some of them can be matched. Find a solution so that you have maximum number of matches.

- The multiple-source max-flow problem.

  Example: A supermarket has several vendors for the same merchandise. It wants to transport the maximum number of merchandise to the market through its own transportation network.

- The multiple-sink max-flow problem.

  Example: A factory wants to send the maximum number of products to several countries through its own transportation network.

- The multiple-source multiple-sink max-flow problem.

- Maximum bipartite matching.

- Many other applications.