Bicubic Interpolation

Electrical and Computer Engineering
McMaster University, Canada

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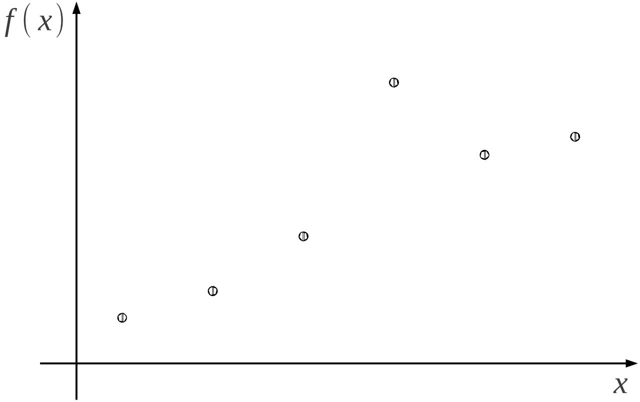
Interpolation

Definition

Interpolation is a method of constructing new data points within the range of a discrete set of known data points.



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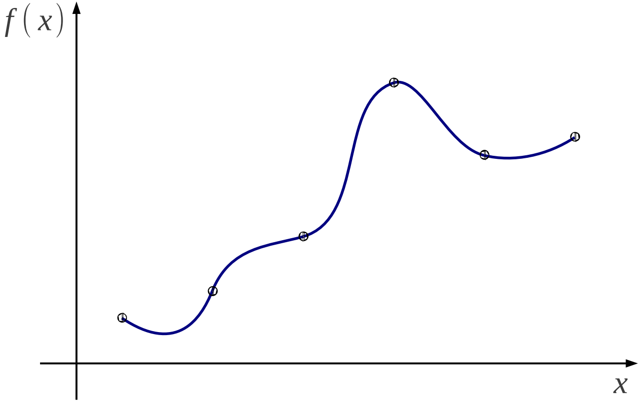
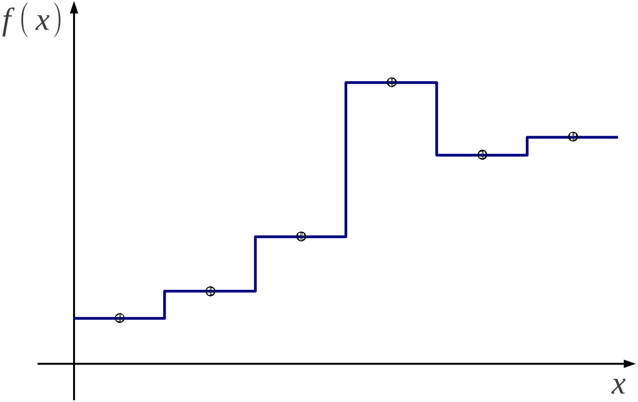


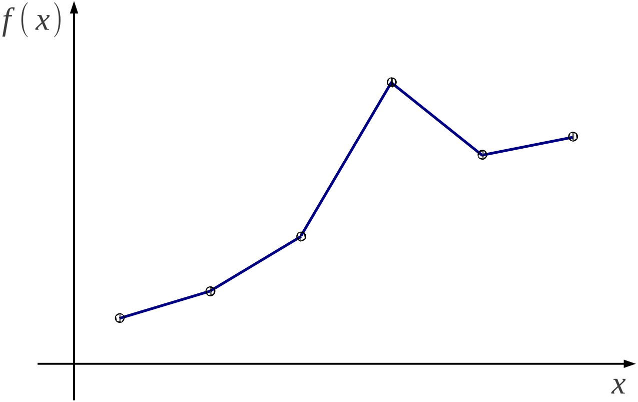
Nearest-neighbour Interpolation

- Use the value of nearest point
- Piecewise-constant function

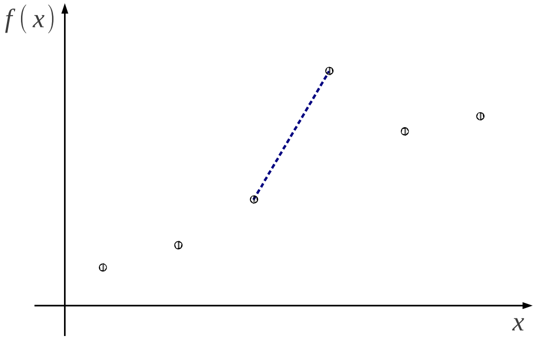


Linear Interpolation

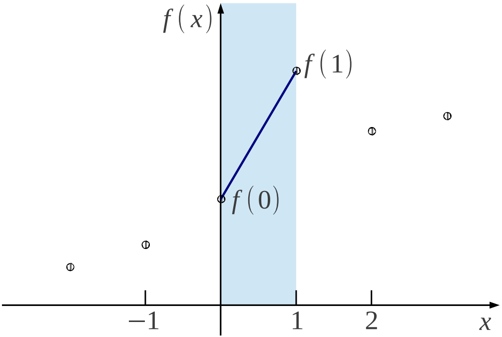
- Straight line between neighbouring points
- Piecewise-linear function



Linear Interpolation

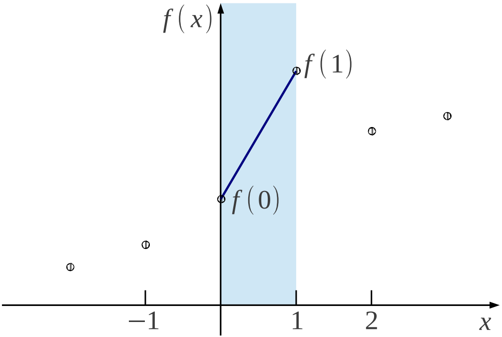


Linear Interpolation



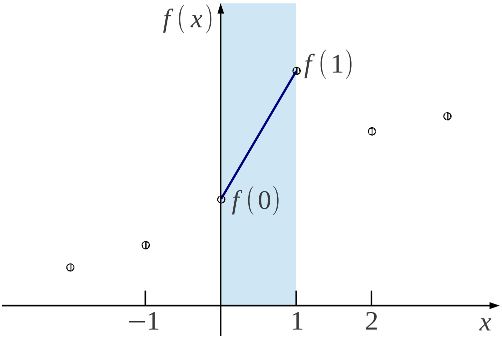
- Normalization

Linear Interpolation



- Normalization
- Model: $f(x) = a_1 x^1 + a_0 x^0$

Linear Interpolation



- Normalization
- Model: $f(x) = a_1 x^1 + a_0 x^0$
- Solve: a_0, a_1

$$\begin{cases} f(0) = a_1 \cdot 0 + a_0 \cdot 1 \\ f(1) = a_1 \cdot 1 + a_0 \cdot 1 \end{cases}$$

Linear Interpolation

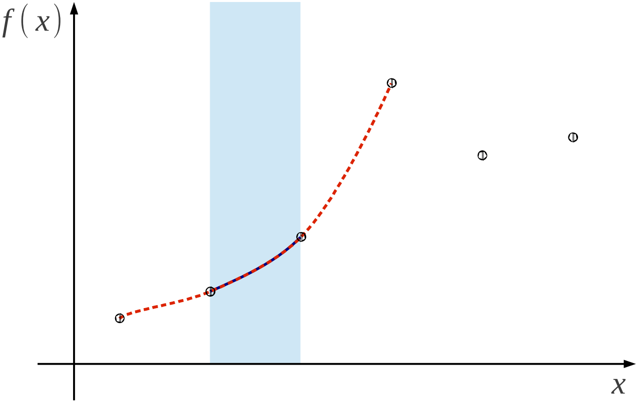
$$\begin{cases} f(0) = a_1 \cdot 0 + a_0 \cdot 1 \\ f(1) = a_1 \cdot 1 + a_0 \cdot 1 \end{cases}$$

- Let $\mathbf{y} = \begin{bmatrix} f(0) & f(1) \end{bmatrix}^{\mathrm{T}}$, $\mathbf{B} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ and $\mathbf{a} = \begin{bmatrix} a_1 & a_0 \end{bmatrix}^{\mathrm{T}}$
- Then the equations can be written as $\mathbf{y} = \mathbf{Ba}$
- Thus $f(x) = \mathbf{ba} = \mathbf{bB}^{-1}\mathbf{y}$, where $\mathbf{b} = \begin{bmatrix} x^1 & x^0 \end{bmatrix}$
- Example:

$$\begin{aligned} f(0.5) &= \begin{bmatrix} 0.5^1 & 0.5^0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^{-1} \mathbf{y} \\ &= \begin{bmatrix} 0.5 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \mathbf{y} \\ &= \begin{bmatrix} 0.5 & 0.5 \end{bmatrix} \mathbf{y} \\ &= \frac{1}{2} f(0) + \frac{1}{2} f(1) \end{aligned}$$

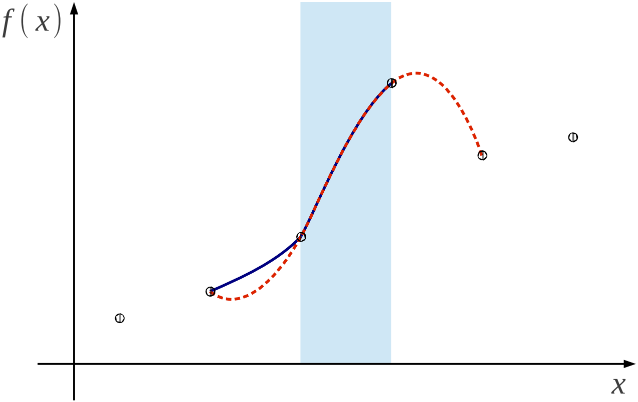
Cubic Interpolation

- Piecewise-cubic function



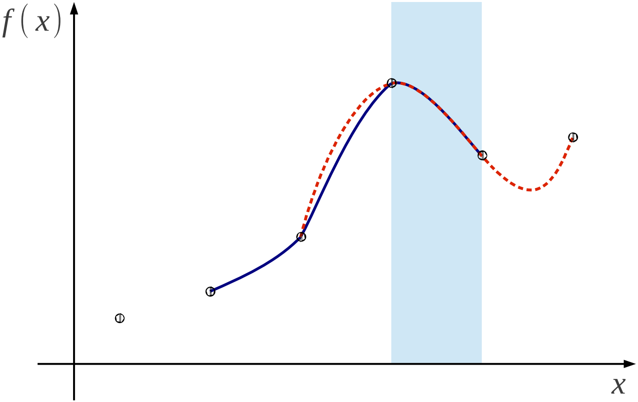
Cubic Interpolation

- Piecewise-cubic function



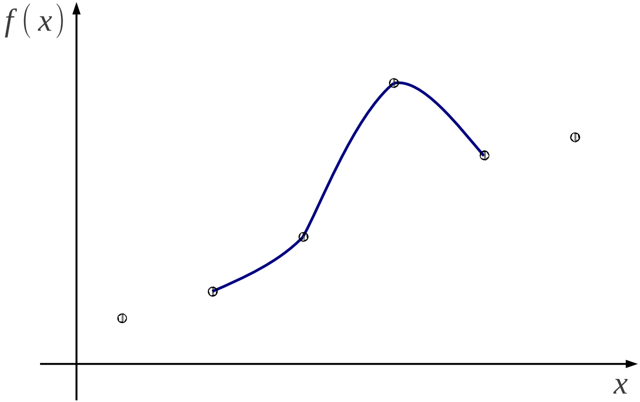
Cubic Interpolation

- Piecewise-cubic function

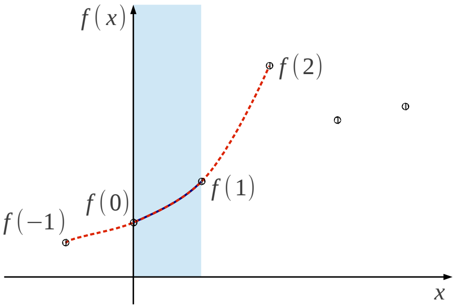


Cubic Interpolation

- Piecewise-cubic function



Cubic Interpolation



- Model: $f(x) = \sum_{i=0}^{3} a_i x^i = a_3 x^3 + a_2 x^2 + a_1 x^1 + a_0$
- $$\begin{cases} f(-1) = a_3 \cdot (-1)^3 + a_2 \cdot (-1)^2 + a_1 \cdot (-1)^1 + a_0 \cdot (-1)^0 \\ f(0) = a_3 \cdot 0^3 + a_2 \cdot 0^2 + a_1 \cdot 0^1 + a_0 \cdot 0^0 \\ f(1) = a_3 \cdot 1^3 + a_2 \cdot 1^2 + a_1 \cdot 1^1 + a_0 \cdot 1^0 \\ f(2) = a_3 \cdot 2^3 + a_2 \cdot 2^2 + a_1 \cdot 2^1 + a_0 \cdot 2^0 \end{cases}$$

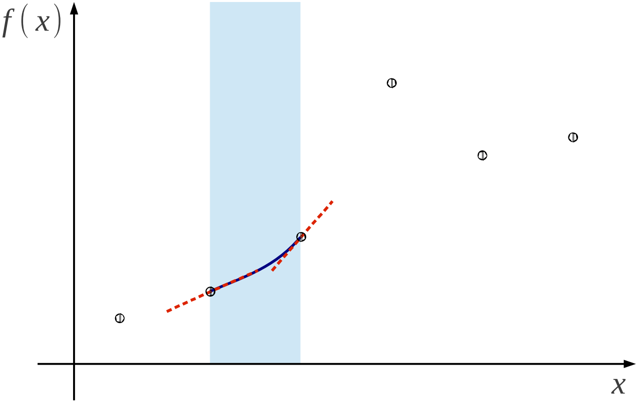
Cubic Interpolation

- Let
  - $\mathbf{y} = \begin{bmatrix} f(-1) & f(0) & f(1) & f(2) \end{bmatrix}^{\mathrm{T}}$
  - $\mathbf{B} = \begin{bmatrix} (-1)^3 & (-1)^2 & (-1)^1 & (-1)^0 \\ 0^3 & 0^2 & 0^1 & 0^0 \\ 1^3 & 1^2 & 1^1 & 1^0 \\ 2^3 & 2^2 & 2^1 & 2^0 \end{bmatrix} = \begin{bmatrix} -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 8 & 4 & 2 & 1 \end{bmatrix}$
  - $\mathbf{a} = \begin{bmatrix} a_3 & a_2 & a_1 & a_0 \end{bmatrix}^{\mathrm{T}}$
- Then the equations can be written as $\mathbf{y} = \mathbf{Ba}$
- Thus $f(x) = \mathbf{ba} = \mathbf{bB}^{-1}\mathbf{y}$, where $\mathbf{b} = \begin{bmatrix} x^3 & x^2 & x^1 & x^0 \end{bmatrix}$
- Example:

$$f(0.5) = \begin{bmatrix} 0.5^3 & 0.5^2 & 0.5^1 & 0.5^0 \end{bmatrix} \begin{bmatrix} -0.167 & 0.5 & -0.5 & 0.167 \\ 0.5 & -1 & 0.5 & 0 \\ -0.333 & -0.5 & 1 & -0.167 \\ 0 & 1 & 0 & 0 \end{bmatrix} \mathbf{y}$$
$$= \begin{bmatrix} -0.0625 & 0.5625 & 0.5625 & -0.0625 \end{bmatrix} \mathbf{y}$$
$$= \frac{1}{16} \begin{bmatrix} -1 & 9 & 9 & -1 \end{bmatrix} \mathbf{y}$$

Cubic Spline Interpolation

- Piecewise-cubic function



Cubic Spline Interpolation

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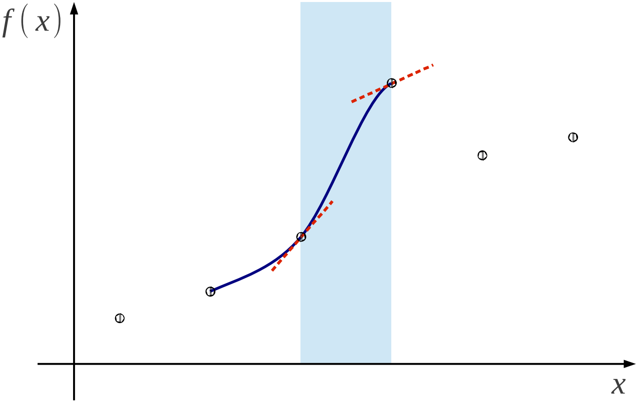
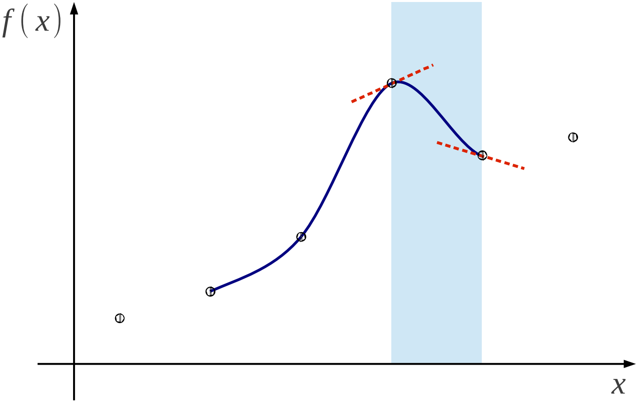


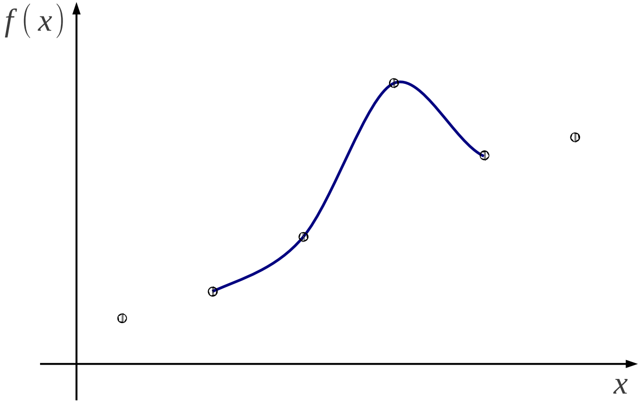
Cubic Spline Interpolation

- Piecewise-cubic function



Cubic Spline Interpolation

- Piecewise-cubic function



Cubic Spline Interpolation

- Model:
  - $f(x) = \sum_{i=0}^{3} a_i x^i = a_3 x^3 + a_2 x^2 + a_1 x^1 + a_0$
  - $f'(x) = \sum_{i=1}^{3} i a_i x^{i-1} = 3a_3 x^2 + 2a_2 x^1 + a_1$
- $$\begin{cases} f(0) = a_3 \cdot 0^3 & + a_2 \cdot 0^2 & + a_1 \cdot 0^1 & + a_0 \cdot 0^0 \\ f(1) = a_3 \cdot 1^3 & + a_2 \cdot 1^2 & + a_1 \cdot 1^1 & + a_0 \cdot 1^0 \\ f'(0) = a_3 \cdot 3 \cdot 0^2 & + a_2 \cdot 2 \cdot 0^1 & + a_1 \cdot 1 \cdot 0^0 \\ f'(1) = a_3 \cdot 3 \cdot 1^2 & + a_2 \cdot 2 \cdot 1^1 & + a_1 \cdot 1 \cdot 1^0 \end{cases}$$
- $$\begin{cases} f(0) = f(0) \\ f(1) = f(1) \\ f'(0) \approx \frac{1}{2} f(1) - \frac{1}{2} f(-1) \\ f'(1) \approx \frac{1}{2} f(2) - \frac{1}{2} f(0) \end{cases}$$

Cubic Spline Interpolation

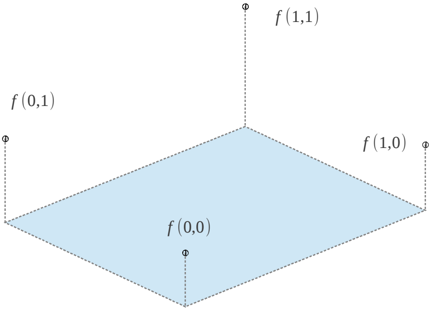
- Let
  - $\mathbf{z} = \begin{bmatrix} f(0) & f(1) & f'(0) & f'(1) \end{bmatrix}^{\mathrm{T}}$
  - $\mathbf{B} = \begin{bmatrix} 0^3 & 0^2 & 0^1 & 0^0 \\ 1^3 & 1^2 & 1^1 & 1^0 \\ 3 \cdot 0^3 & 2 \cdot 0^2 & 1 \cdot 0^1 & 0 \\ 3 \cdot 1^3 & 2 \cdot 1^2 & 1 \cdot 1^1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix}$
  - $\mathbf{a} = \begin{bmatrix} a_3 & a_2 & a_1 & a_0 \end{bmatrix}^{\mathrm{T}}$
- Then the first set of equations can be written as $\mathbf{z} = \mathbf{Ba}$
- Let
  - $\mathbf{y} = \begin{bmatrix} f(-1) & f(0) & f(1) & f(2) \end{bmatrix}^{\mathrm{T}}$
  - $\mathbf{C} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$
- Then the second set of equations can be written as $\mathbf{z} = \mathbf{Cy}$
- Thus $\mathbf{Ba} = \mathbf{Cy}$, and $\mathbf{a} = \mathbf{B}^{-1}\mathbf{Cy}$

Cubic Spline Interpolation

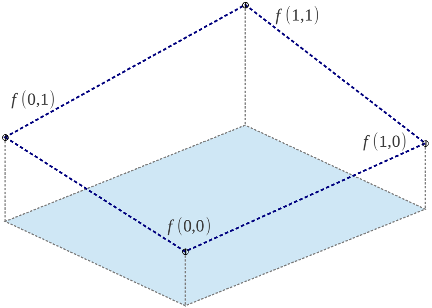
- $f(x) = \mathbf{ba} = \mathbf{bB}^{-1}\mathbf{Cy}$, where $\mathbf{b} = \begin{bmatrix} x^3 & x^2 & x^1 & x^0 \end{bmatrix}$
- Example:

$$
\begin{aligned}
f(0.5) &= \begin{bmatrix} 0.5^3 & 0.5^2 & 0.5^1 & 0.5^0 \end{bmatrix} (\mathbf{B}^{-1}\mathbf{C})\mathbf{y} \\
&= \begin{bmatrix} 0.125 & 0.25 & 0.5 & 1 \end{bmatrix} \begin{bmatrix} -0.5 & 1.5 & -1.5 & 0.5 \\ 1 & -2.5 & 2 & -0.5 \\ -0.5 & 0 & 0.5 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \mathbf{y} \\
&= \begin{bmatrix} -0.0625 & 0.5625 & 0.5625 & -0.0625 \end{bmatrix} \mathbf{y} \\
&= \frac{1}{16} \begin{bmatrix} -1 & 9 & 9 & -1 \end{bmatrix} \mathbf{y}
\end{aligned}
$$

Bilinear Interpolation

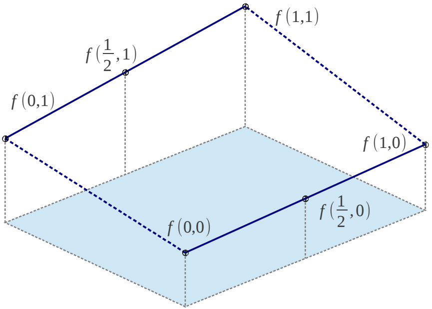


Bilinear Interpolation



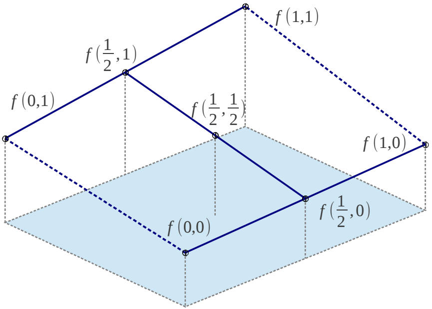
- Model $f(x, y)$ as a bilinear surface

Bilinear Interpolation



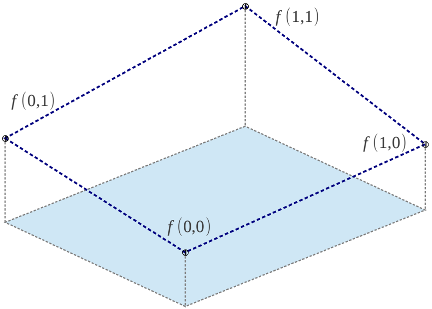
- Model $f(x, y)$ as a bilinear surface
- Interpolate $f(\frac{1}{2}, 0)$ using $f(0, 0)$ and $f(1, 0)$
  Interpolate $f(\frac{1}{2}, 1)$ using $f(0, 1)$ and $f(1, 1)$

Bilinear Interpolation



- Model $f(x, y)$ as a bilinear surface
- Interpolate $f(\frac{1}{2}, 0)$ using $f(0, 0)$ and $f(1, 0)$
  Interpolate $f(\frac{1}{2}, 1)$ using $f(0, 1)$ and $f(1, 1)$
- Interpolate $f(\frac{1}{2}, \frac{1}{2})$ using $f(\frac{1}{2}, 0)$ and $f(\frac{1}{2}, 1)$

Bilinear Interpolation



- Model: $f(x,y) = \sum_{i=0}^{1}\sum_{i=0}^{1} a_{ij}x^iy^i = a_{11}xy + a_{10}x + a_{01}y + a_{00}$
- $\begin{cases} f(0,0) = a_{11}\cdot 0 + a_{10}\cdot 0 + a_{01}\cdot 0 + a_{00}\cdot 1 \\ f(0,1) = a_{11}\cdot 0 + a_{10}\cdot 0 + a_{01}\cdot 1 + a_{00}\cdot 1 \\ f(1,0) = a_{11}\cdot 0 + a_{10}\cdot 1 + a_{01}\cdot 0 + a_{00}\cdot 1 \\ f(1,1) = a_{11}\cdot 1 + a_{10}\cdot 1 + a_{01}\cdot 1 + a_{00}\cdot 1 \end{cases}$

Bicubic Interpolation

Model: $f(x,y) = \sum_{i=0}^{3}\sum_{j=0}^{3} a_{ij}x^i y^j$

$$f(x,y) = \begin{bmatrix} x^3 & x^2 & x & 1 \end{bmatrix} \begin{bmatrix} a_{3,3} & a_{3,2} & a_{3,1} & a_{3,0} \\ a_{2,3} & a_{2,2} & a_{2,1} & a_{2,0} \\ a_{1,3} & a_{1,2} & a_{1,1} & a_{1,0} \\ a_{0,3} & a_{0,2} & a_{0,1} & a_{0,0} \end{bmatrix} \begin{bmatrix} y^3 \\ y^2 \\ y \\ 1 \end{bmatrix}$$

Given 4×4 known data points, we have $F = BAB^{\mathrm{T}}$

$$\begin{bmatrix} f(-1,-1) & f(-1,0) & f(-1,1) & f(-1,2) \\ f(0,-1) & f(0,0) & f(0,1) & f(0,2) \\ f(1,-1) & f(1,0) & f(1,1) & f(1,2) \\ f(2,-1) & f(2,0) & f(2,1) & f(2,2) \end{bmatrix}_F$$

$$= \begin{bmatrix} (-1)^3 & (-1)^2 & -1 & 1 \\ 0^3 & 0^2 & 0 & 1 \\ 1^3 & 1^2 & 1 & 1 \\ 2^3 & 2^2 & 2 & 1 \end{bmatrix}_B \begin{bmatrix} a_{3,3} & a_{3,2} & a_{3,1} & a_{3,0} \\ a_{2,3} & a_{2,2} & a_{2,1} & a_{2,0} \\ a_{1,3} & a_{1,2} & a_{1,1} & a_{1,0} \\ a_{0,3} & a_{0,2} & a_{0,1} & a_{0,0} \end{bmatrix}_A \begin{bmatrix} (-1)^3 & 0^3 & 1^3 & 2^3 \\ (-1)^2 & 0^2 & 1^2 & 2^2 \\ -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix}_{B^T}$$

Bicubic Interpolation(cont.)

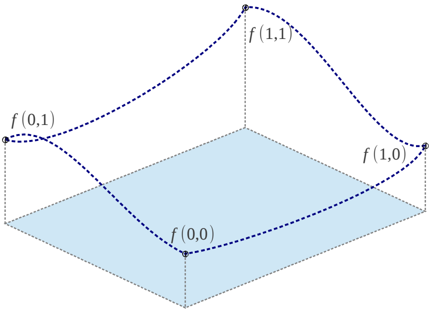
$$F = BAB^{\mathrm{T}} \Rrightarrow A = B^{-1}F(B^{\mathrm{T}})^{-1} = B^{-1}F(B^{-1})^{\mathrm{T}}$$

$$B = \begin{bmatrix} -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 8 & 4 & 2 & 1 \end{bmatrix}; \quad B^{-1} = \begin{bmatrix} -1/6 & 1/2 & -1/2 & 1/6 \\ 1/2 & -1 & 1/2 & 0 \\ -1/3 & -1/2 & 1 & -1/6 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

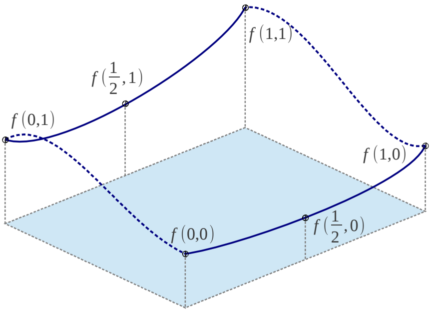
$$f(\frac{1}{2}, \frac{1}{2}) = \begin{bmatrix} \frac{1}{8} & \frac{1}{4} & \frac{1}{2} & 1 \end{bmatrix}_x B^{-1}F(B^{-1})^{\mathrm{T}} \begin{bmatrix} \frac{1}{8} & \frac{1}{4} & \frac{1}{2} & 1 \end{bmatrix}_y^{\mathrm{T}}$$

$$= \begin{bmatrix} \frac{-1}{16} & \frac{9}{16} & \frac{9}{16} & \frac{-1}{16} \end{bmatrix} \begin{bmatrix} f(-1,-1) & f(-1,0) & f(-1,1) & f(-1,2) \\ f(0,-1) & f(0,0) & f(0,1) & f(0,2) \\ f(1,-1) & f(1,0) & f(1,1) & f(1,2) \\ f(2,-1) & f(2,0) & f(2,1) & f(2,2) \end{bmatrix} \begin{bmatrix} \frac{-1}{16} \\ \frac{9}{16} \\ \frac{9}{16} \\ \frac{-1}{16} \end{bmatrix}$$

Bicubic Spline Interpolation

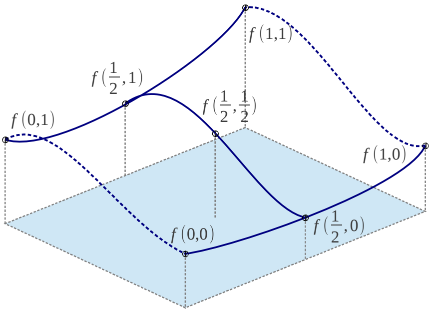


Bicubic Spline Interpolation



- Interpolate
  - $f(\frac{1}{2},0)$ using $f(0,0)$, $f(1,0)$, $\partial_x f(0,0)$ and $\partial_x f(1,0)$
  - $f(\frac{1}{2},1)$ using $f(0,1)$, $f(1,1)$, $\partial_x f(0,1)$ and $\partial_x f(1,1)$
  - $\partial_y f(\frac{1}{2},0)$ using $\partial_y f(0,0)$, $\partial_y f(1,0)$, $\partial_{xy} f(0,0)$ and $\partial_{xy} f(1,0)$
  - $\partial_y f(\frac{1}{2},1)$ using $\partial_y f(0,1)$, $\partial_y f(1,1)$, $\partial_{xy} f(0,1)$ and $\partial_{xy} f(1,1)$

Bicubic Spline Interpolation



- Interpolate
  - $f(\frac{1}{2},0)$ using $f(0,0)$, $f(1,0)$, $\partial_x f(0,0)$ and $\partial_x f(1,0)$
  - $f(\frac{1}{2},1)$ using $f(0,1)$, $f(1,1)$, $\partial_x f(0,1)$ and $\partial_x f(1,1)$
  - $\partial_y f(\frac{1}{2},0)$ using $\partial_y f(0,0)$, $\partial_y f(1,0)$, $\partial_{xy} f(0,0)$ and $\partial_{xy} f(1,0)$
  - $\partial_y f(\frac{1}{2},1)$ using $\partial_y f(0,1)$, $\partial_y f(1,1)$, $\partial_{xy} f(0,1)$ and $\partial_{xy} f(1,1)$
- Interpolate $f(\frac{1}{2},\frac{1}{2})$ using $f(\frac{1}{2},0)$, $f(\frac{1}{2},1)$, $\partial_y f(\frac{1}{2},0)$ and $\partial_y f(\frac{1}{2},1)$

Bicubic Spline Interpolation

- Model:
  - $f(x, y) = \sum_{i=0}^{3} \sum_{j=0}^{3} a_{ij} x^i y^j$
  - $\partial_x f(x, y) = \sum_{i=1}^{3} \sum_{j=0}^{3} i a_{ij} x^{i-1} y^j$
  - $\partial_y f(x, y) = \sum_{i=0}^{3} \sum_{j=1}^{3} j a_{ij} x^i y^{j-1}$
  - $\partial_{xy} f(x, y) = \sum_{i=1}^{3} \sum_{j=1}^{3} ij a_{ij} x^{i-1} y^{j-1}$
- Approximation:
  - $\partial_x f(x, y) = [f(x+1, y) - f(x-1, y)]/2$
  - $\partial_y f(x, y) = [f(x, y+1) - f(x, y-1)]/2$
  - $\partial_{xy} f(x, y) = [f(x+1, y+1) - f(x-1, y) - f(x, y-1) + f(x, y)]/4$

Examples

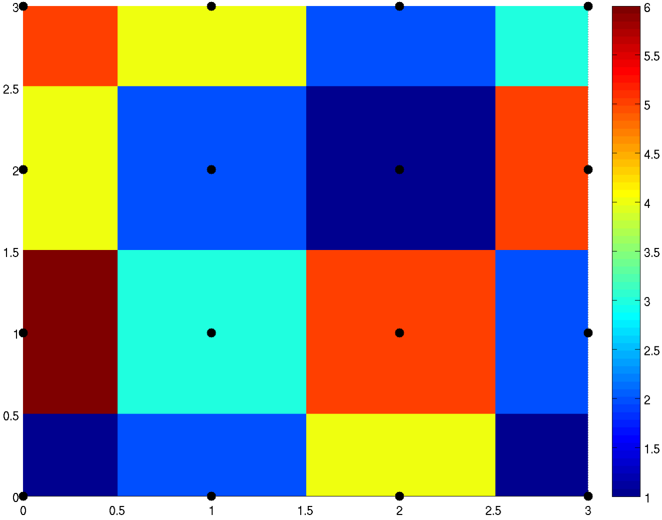


Figure : Nearest Neighbour Interpolation

Examples

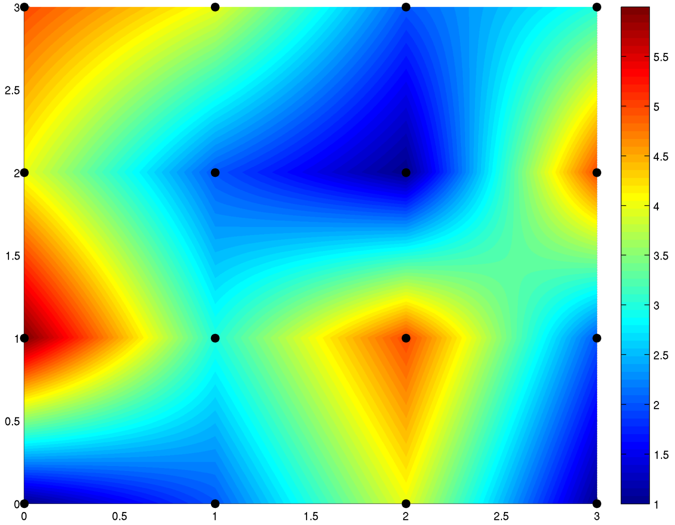


Figure : Bilinear Interpolation

Examples

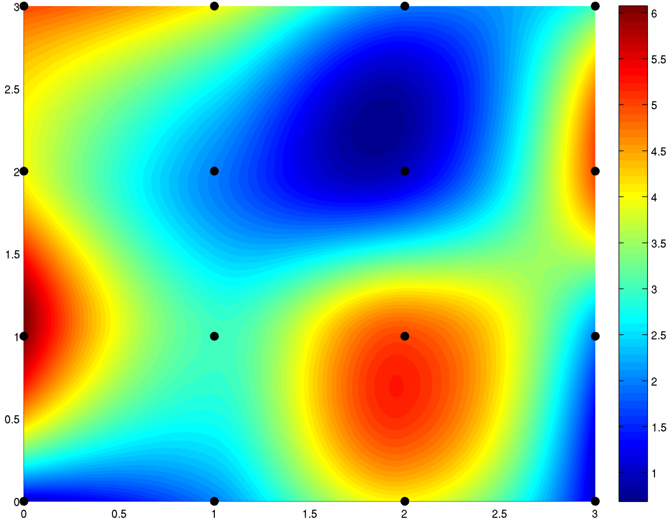


Figure : Bicubic Spline Interpolation

Examples

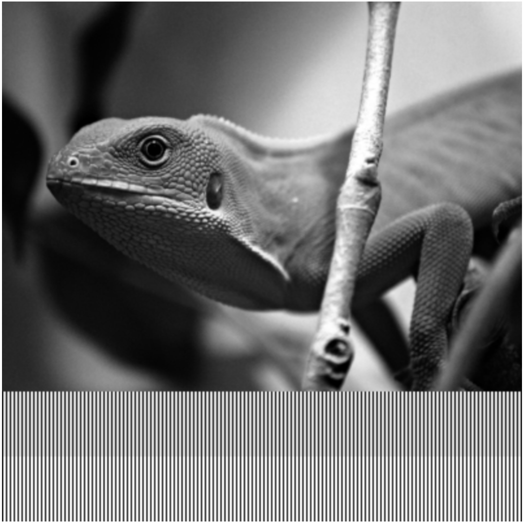


Figure : Bilinear Interpolation

Examples

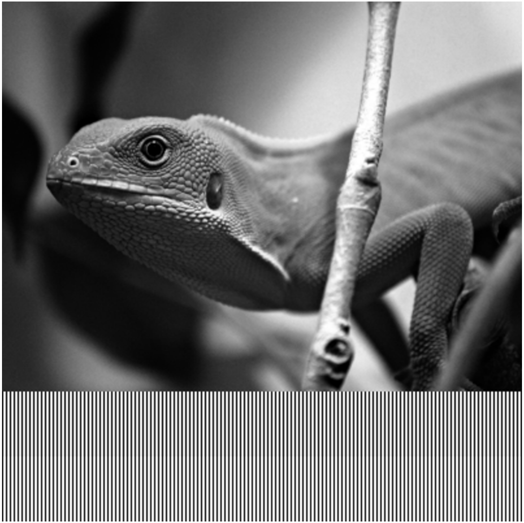


Figure : Bicubic Interpolation

Properties of Linear and Cubic Interpolations

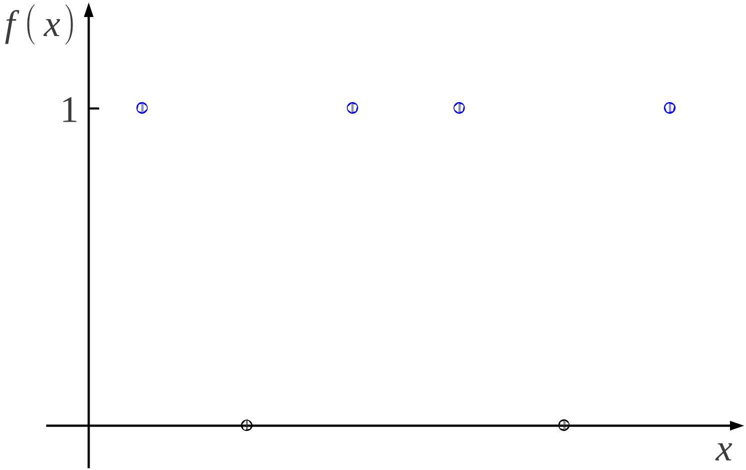
- Linear: $f_l(0.5) = \frac{1}{2}f(0) + \frac{1}{2}f(1)$

  Cubic: $f_c(0.5) = -\frac{1}{16}(-1) + \frac{9}{16}f(0) + \frac{9}{16}f(1) - \frac{1}{16}f(2)$

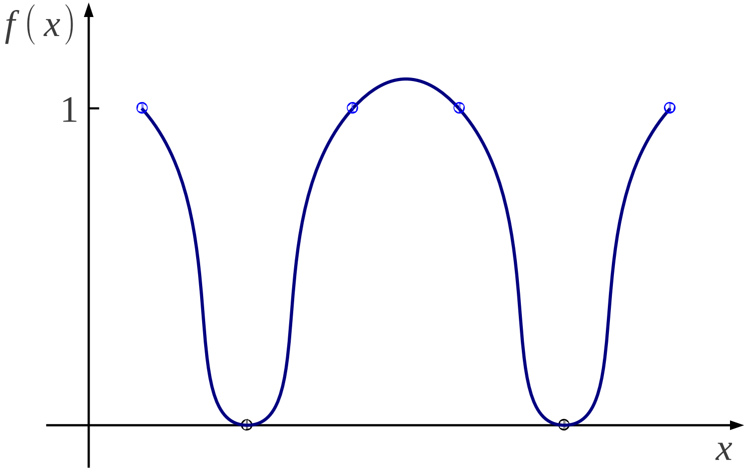
- The absolute difference between the results of linear and cubic interpolations

$$
\begin{aligned}
& |f_c(0.5) - f_l(0.5)| \\
= & \left|-\frac{1}{16}f(-1) + \frac{1}{16}f(0) + \frac{1}{16}f(1) - \frac{1}{16}f(2)\right| \\
\leq & \frac{|-0+1+1-0|}{16} \\
= & 0.125
\end{aligned}
$$

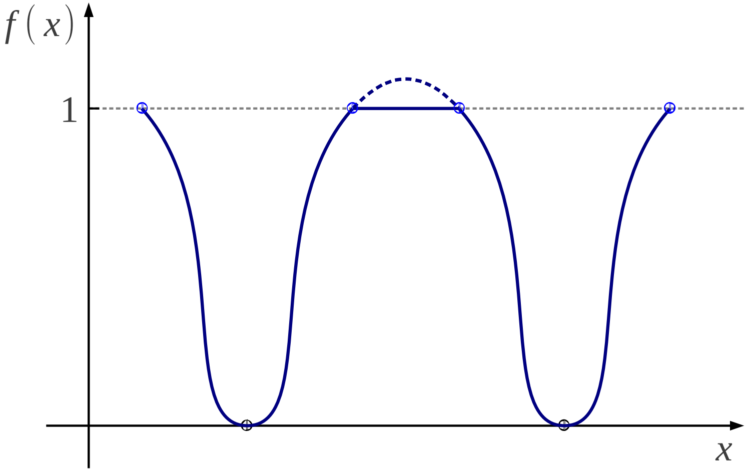
Properties of Linear and Cubic Interpolations



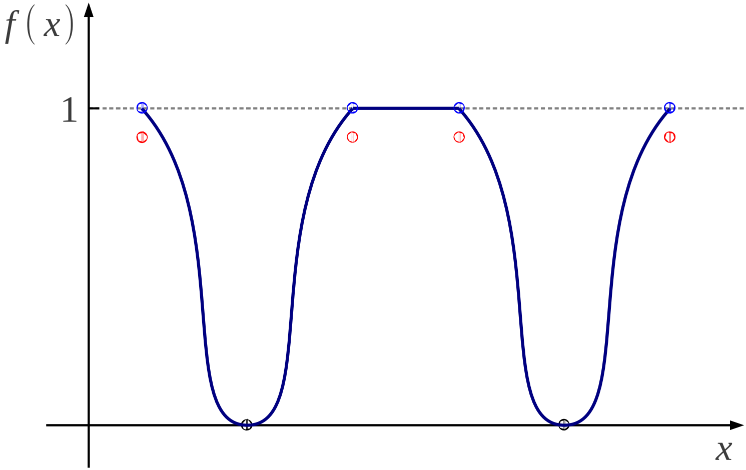
Properties of Linear and Cubic Interpolations



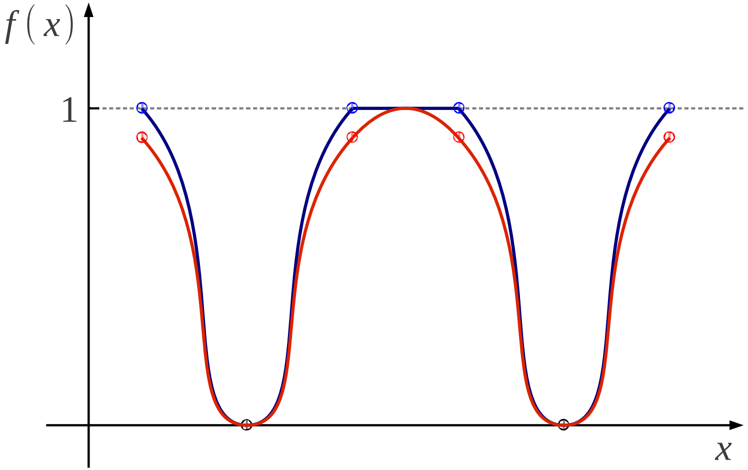
Properties of Linear and Cubic Interpolations



Properties of Linear and Cubic Interpolations



Properties of Linear and Cubic Interpolations



Properties of Linear and Cubic Interpolations