Chapter 7: Ordinary Differential Equations

Given $\frac{dy}{dx} = f(x, y)$, find $y(x)$.

1 Analytical Method

Given $\frac{dy}{dx} + ky = f(x)$ and initial condition (x_0, y_0)

- Step 1: Find a particular solution, y_p
  - If $f(x) = x$, then $y_p = Ax + B$.
  - If $f(x) = x^2$, then $y_p = Ax^2 + Bx + C$
  - If $f(x) = \sin \omega x$ or $\cos \omega x$, then $y_p = A \sin \omega x + B \cos \omega x$.
  - If $f(x) = e^{rx}$, $r \neq -k$, then $y_p = Ae^{rx}$.
  - If $f(x) = e^{-kx}$, then $y_p = Axe^{-kx}$.
- Step 2: Find the general solution of the homogeneous differential equation $\frac{dy}{dx} + ky = 0$
  $\frac{dy}{ky} = -dx, \ \rightarrow \ \int \frac{dy}{ky} = \int -dx, \ \rightarrow \ \ln y = -kx + ck \ \rightarrow \ y = e^{-kx}e^{ck}$, or
  $y_h = Ce^{-kx}$
- Find constant C using initial condition (x_0, y_0)

2 Euler's Method

Basic idea of iterative methods: given (x_i, y_i), $x_{i+1} = x_i + h$, $y_{i+1} = y_i + \phi h$, where ϕ is estimated function slope.

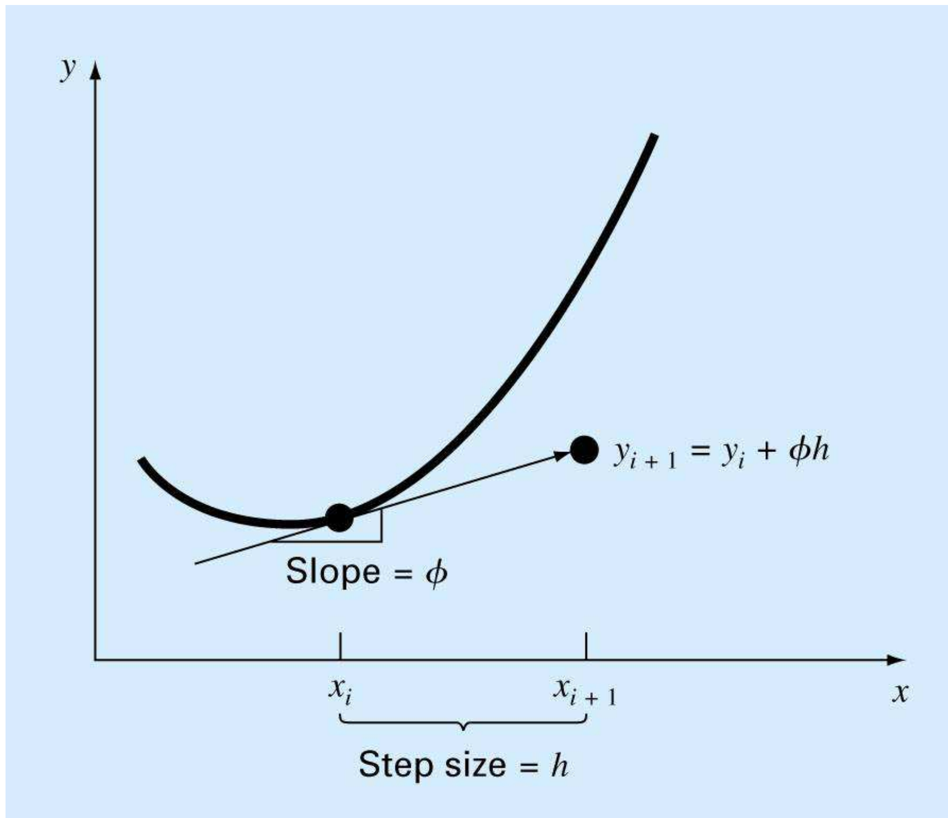

Figure 1: Illustration of iterative methods

In Euler's method, the first derivative is used to estimate the function slope, i.e., $\phi = f(x_i, y_i)$, and $y_{i+1} = y_i + f(x_i, y_i) \cdot h$.

Using Taylor serious to analyze local truncation error

If $y(x)$ is continuous and its derivatives are continuous too, its Taylor series can be represented as

$$y_{i+1} = y_i + y_i^{'}h + \frac{y_i^{''}}{2!}h^2 + \cdots + \frac{y_i^{(n)}}{n!}h^n + R_n$$

where $h = x_{i+1} - x_i$ and R_n is the remainder term given by

$$R_n = \frac{y^{(n+1)}(\alpha)}{(n+1)!}h^{n+1} = O(h^{n+1})$$

and α is a value between x_i and x_{i+1}. Since $y^{'} = \frac{dy}{dx} = f(x, y)$, we have $y_i^{'} = f(x_i, y_i)$, $y_i^{''} = f^{'}(x_i, y_i)$, $\ldots$, and $y_i^{(n)} = f^{(n-1)}(x_i, y_i)$. Then

$$y_{i+1} = y_i + f(x_i, y_i)h + \frac{1}{2}f^{'}(x_i, y_i)h^2 + \cdots + \frac{1}{n!}f^{(n-1)}(x_i, y_i) + O(h^{n+1})$$

Using Eular's method,

$$y_{i+1} = y_i + f(x_i, y_i)h$$

Therefore, the true local truncation error in using Euler's method is

$$E_t = \frac{1}{2}f^{'}(x_i, y_i)h^2 + \cdots + \frac{1}{n!}f^{(n-1)}(x_i, y_i) + O(h^{n+1})$$

When h is sufficiently small, the higher order terms can be neglected, and the approximated local truncation error is

$$E_a = \frac{1}{2} f^{'}(x_i, y_i) h^2$$

- Local absolute truncation error, E_a, is proportional to h^2 and $f^{'}(x_i, y_i)$.
- Taylor series only provides the local truncation error.
- Global truncation error using Euler's method is proportional to the step size, $O(h)$.
- The truncation error can be reduced by decreasing the step size.
- Euler's method provides error free prediction if the function $y(x)$ is linear.

Example: Integrate the equation $\frac{dy}{dx} = -2x^3 + 12x^2 - 20x + 8.5$ from $x = 0$ to $x = 1$ (1) using analytical method, and (2) using Euler's method with a step size of 0.5 and 0.25. The initial condition at $x = 0$ is $y = 1$.

Solution:

$f(x, y) = -2x^3 + 12x^2 - 20x + 8.5$, $x_0 = 0$, and $y_0 = 1$.

Using analytical method: The exact solution to the equation $\frac{dy}{dx} = -2x^3 + 12x^2 - 20x + 8.5$ is

$$y = \int (-2x^3 + 12x^2 - 20x + 8.5) dx = -\frac{1}{2} x^4 + 4x^3 - 10x^2 + 8.5x + C$$

where C is a constant. Using the initial condition $y = 1$ when $x = 0$, then $1 = C$. Thus,

$$y = -\frac{1}{2}x^4 + 4x^3 - 10x^2 + 8.5x + 1$$

When $x = 0.5$, the true function value is

$$y(0.5) = -\frac{1}{2} \times 0.5^4 + 4 \times 0.5^3 - 10 \times 0.5^2 + 8.5 \times 0.5 + 1 = 3.21875$$

and when $x = 1$, the true function value is

$$y(1) = -\frac{1}{2} \times 1^4 + 4 \times 1^3 - 10 \times 1^2 + 8.5 \times 1 + 1 = 3$$

Using Euler's method with $h = 0.5$:

$x_1 = x_0 + h = 0.5$, and $y_1 = y_0 + f(x_0, y_0)h = 1 + f(0, 1) \times 0.5 = 1 + 8.5 \times 0.5 = 5.25$.
The percent relative error is

$$\epsilon_t = |\frac{\text{true value} - \text{approximate}}{\text{true value}}| \times 100\% = [(3.21875 - 5.25)/3.21875] \times 100\% = 63.1\%$$

$x_2 = x_1 + h = 0.5 + 0.5 = 1$, and $y_2 = y_1 + f(x_1, y_1)h = 5.25 + f(0.5, 5.25) \times 0.5 = 5.875$.
The percent relative error is $\epsilon_t = [(3 - 5.875)/3] \times 100\% = 95.8\%$.

Using Euler's method with $h = 0.25$:

$x_1 = x_0 + h = 0.25$, and $y_1 = y_0 + f(x_0, y_0)h = 1 + f(0, 1) \times 0.25 = 1 + 8.5 \times 0.25 = 3.1250$.

$x_2 = x_1 + h = 0.25 + 0.25 = 0.5$, and $y_2 = y_1 + f(x_1, y_1)h = 3.1250 + f(0.25, 3.1250) \times 0.25 = 3.1250 + 4.2188 \times 0.25 = 4.1797$

The percent relative error is

$$\epsilon_t = |\frac{\text{true value} - \text{approximate}}{\text{true value}}| \times 100\% = [\frac{3.21875 - 4.1797}{3.21875} \times 100\% = 29.85\%$$

$x_3 = x_2 + h = 0.5 + 0.25 = 0.75$, and $y_3 = y_2 + f(x_2, y_2)h = 4.1797 + f(0.5, 4.1797) \times 0.25 = 4.4922$.

$x_4 = x_3 + h = 0.75 + 0.25 = 1$, and $y_4 = y_3 + f(x_3, y_3)h = 4.4922 + f(0.75, 4.4922) \times 0.25 = 4.3438$.

The percent relative error is $\epsilon_t = [(3 - 4.3438)/3] \times 100\% = 44.79\%$.

Reducing step size can reduce the estimation error. Another approach to reducing the estimation error is to use higher order Taylor series.

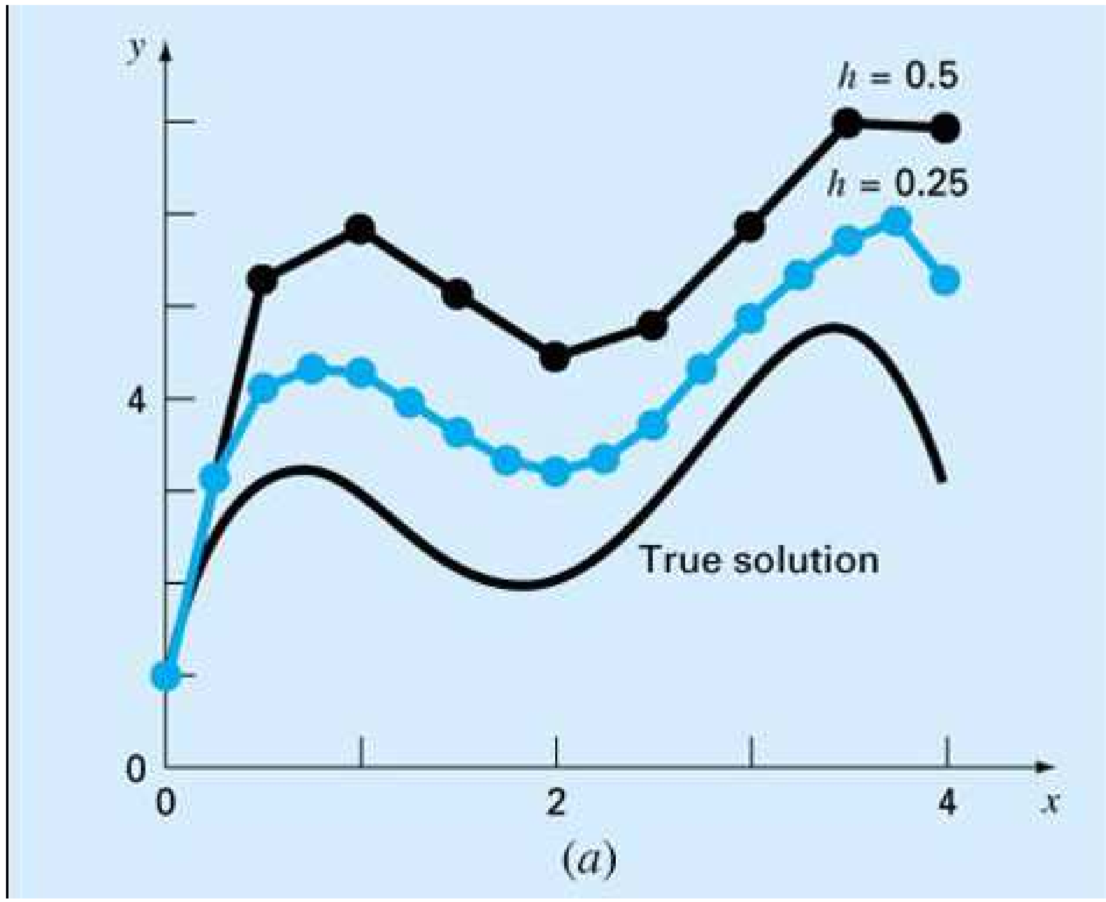

Figure 2: Euler's method for $\frac{dy}{dx} = -2x^3 + 12x^2 - 20x + 8.5$

3 Higher-order Taylor Series Methods

Using the second-order Taylor series,

$$y_{i+1} = y_i + y_i^{'}h + \frac{y_i^{''}}{2!}h^2 = y_i + f(x_i, y_i)h + \frac{1}{2}f^{'}(x_i, y_i)h^2$$

where $f^{'}(x, y)$ is found using the chain-rule as

$$f^{'}(x, y) = \frac{\partial f(x, y)}{\partial x} + \frac{\partial f(x, y)}{\partial y}\frac{dy}{dx}$$

Using this method, the approximate local truncation error is

$$E_a = \frac{1}{3!}f^{''}(x_i, y_i)h^3 = \frac{1}{6}f^{''}(x_i, y_i)h^3$$

where

$$f^{''}(x, y) = \frac{\partial f^{'}(x, y)}{\partial x} + \frac{\partial f^{'}(x, y)}{\partial y}\frac{dy}{dx}$$

$f^{'}(x, y)$ and $f^{''}(x, y)$ may be difficult to evaluate for complicated functions.

4 Runge-Kutta Methods

Runge-Kutta (RK) methods can achieve the accuracy of higher order Taylor series but avoid evaluating the higher order derivatives. The general form of RK methods is

$$y_{i+1} = y_i + \phi(x_i, y_i, h)h$$

where $\phi(x_i, y_i, h)$ is called an increment function and is written in general form as

$$\phi = a_1 k_1 + a_2 k_2 + \cdots + a_n k_n$$

where

$$\begin{aligned}
k_1 &= f(x_i, y_i) \\
k_2 &= f(x_i + p_1 h, y_i + q_{11} k_1 h) \\
k_3 &= f(x_i + p_2 h, y_i + q_{21} k_1 h + q_{22} k_2 h) \\
&\ldots \\
k_n &= f(x_i + p_{n-1} h, y_i + q_{n-1,1} k_1 h + q_{n-1,2} k_2 h + \cdots + q_{n-1,n-1} k_{n-1} h)
\end{aligned}$$

Various types of RK methods can be devised by employing different numbers of terms in ϕ and different values of the parameters a's p's and q's. For lower order versions of RK methods, the number of terms used is same as the order of the approach.

First-order RK methods

When $n = 1$, letting $a_1 = 1$, we have $\phi(x_i, y_i, h) = a_1 k_1 = k_1$. Then

$$y_{i+1} = y_i + f(x_i, y_i)h$$

is Euler's method. That is, Euler's method is the first-order RK method.

Second-order RK methods

The second-order RK methods use

$$y_{i+1} = y_i + (a_1 k_1 + a_2 k_2)h$$

where

$$\begin{aligned} k_1 &= f(x_i, y_i) \\ k_2 &= f(x_i + p_1 h, y_i + q_{11} k_1 h) \end{aligned}$$

How to find constants a_1, a_2, p_1 and q_{11}?

Using Taylor series:

$$\begin{aligned} y_{i+1} &= y_i + y_i^{'} h + \frac{1}{2} y_i^{''} h^2 \quad \text{(ignore higher order terms)} \\ &= y_i + f(x_i, y_i)h + \frac{1}{2} f^{'}(x_i, y_i) h^2 \\ &= y_i + f(x_i, y_i)h + \frac{1}{2}\left[\frac{\partial f(x_i, y_i)}{\partial x} + \frac{\partial f(x_i, y_i)}{\partial y} y_i^{'}\right] h^2 \\ &= y_i + f(x_i, y_i)h + \frac{1}{2}\frac{\partial f(x_i, y_i)}{\partial x} h^2 + \frac{1}{2}\frac{\partial f(x_i, y_i)}{\partial y} y_i^{'} h^2 \end{aligned} \tag{1}$$

Using 2nd-order RK method,

$$\begin{aligned} y_{i+1} &= y_i + a_1 k_1 h + a_2 k_2 h \\ &= y_i + a_1 f(x_i, y_i) h + a_2 k_2 h \end{aligned} \tag{2}$$

where k_2 can be expanded in Taylor series as

$$\begin{aligned} k_2 &= f(x_i + p_1 h, y_i + q_{11} k_1 h) = f(x_i, y_i) + \frac{\partial f(x_i, y_i)}{\partial x} p_1 h + \frac{\partial f(x_i, y_i)}{\partial y} q_{11} k_1 h \\ &\qquad \text{(ignore higher order terms)} \\ &= f(x_i, y_i) + \frac{\partial f(x_i, y_i)}{\partial x} p_1 h + \frac{\partial f(x_i, y_i)}{\partial y} q_{11} f(x_i, y_i) h \end{aligned} \tag{3}$$

Substituting k_2 in (2) by (3), we have

$$\begin{aligned} y_{i+1} &= y_i + a_1 f(x_i, y_i) h + a_2 f(x_i, y_i) h + a_2 \frac{\partial f(x_i, y_i)}{\partial x} p_1 h^2 + a_2 \frac{\partial f(x_i, y_i)}{\partial y} q_{11} f(x_i, y_i) h^2 \\ &= y_i + (a_1 + a_2) f(x_i, y_i) h + a_2 \frac{\partial f(x_i, y_i)}{\partial x} p_1 h^2 + a_2 \frac{\partial f(x_i, y_i)}{\partial y} q_{11} f(x_i, y_i) h^2 \end{aligned} \tag{4}$$

Comparing the like terms in (4) and (1), we have

$$\begin{aligned} a_1 + a_2 &= 1 \\ a_2 p_1 &= \frac{1}{2} \\ a_2 q_{11} &= \frac{1}{2} \end{aligned}$$

There are three simultaneous equations containing four unkown constants. Therefore, there are infinite sets of constants that satisfy the equations. By assuming a value for one of the constants, we can determine the other three.

Heun method: $a_1 = \frac{1}{2}$, $a_2 = \frac{1}{2}$, and $p_1 = q_{11} = 1$. Then

$$
\begin{aligned}
y_{i+1} &= y_i + (a_1 k_1 + a_2 k_2)h = y_i + \frac{1}{2}(k_1 + k_2)h \\
k_1 &= f(x_i, y_i) \\
k_2 &= f(x_i + h, y_i + k_1 h)
\end{aligned}
$$

Predictor: $y^0_{i+1} = y_i + f(x_i, y_i)h$

Corrector: $y_{i+1} = y_i + \frac{f(x_i,y_i)+f(x_{i+1},y^0_{i+1})}{2}h$

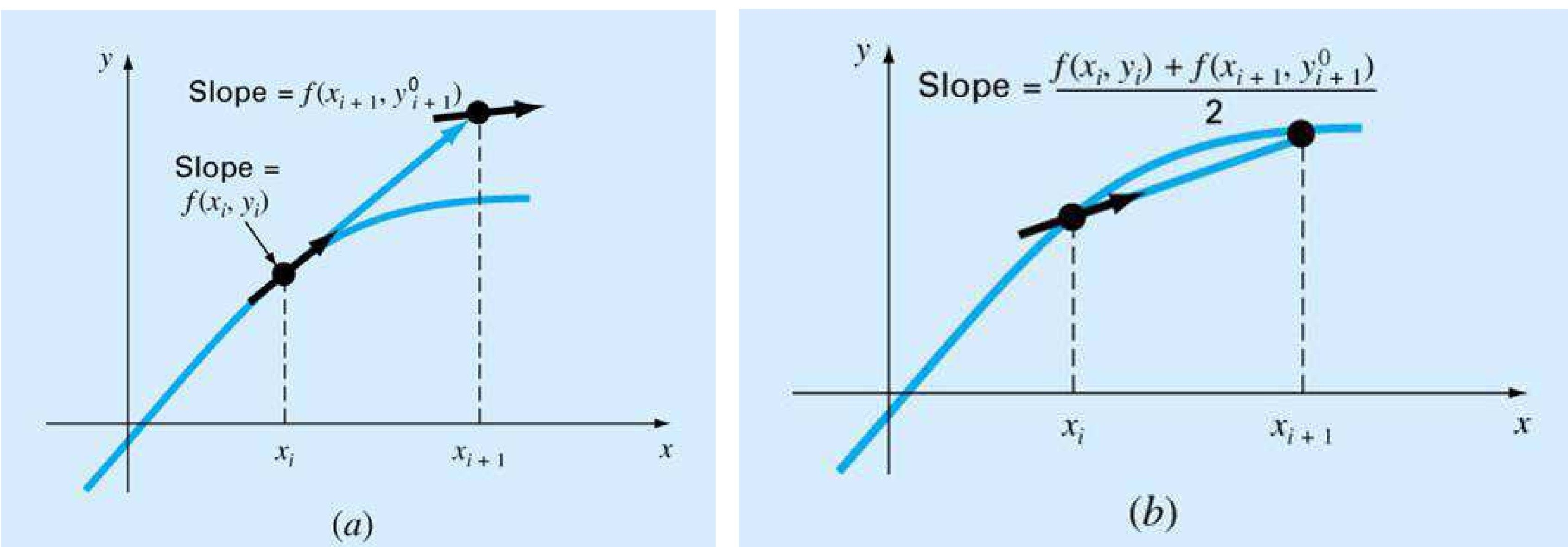

Illustration of Heun's method (a) predictor (b) corrector

The midpoint method $a_1 = 0$, $a_2 = 1$, and $p_1 = q_{11} = \frac{1}{2}$. Then

$$y_{i+1} = y_i + k_2 h$$
$$k_1 = f(x_i, y_i)$$
$$k_2 = f(x_i + \frac{h}{2}, y_i + \frac{k_1 h}{2})$$

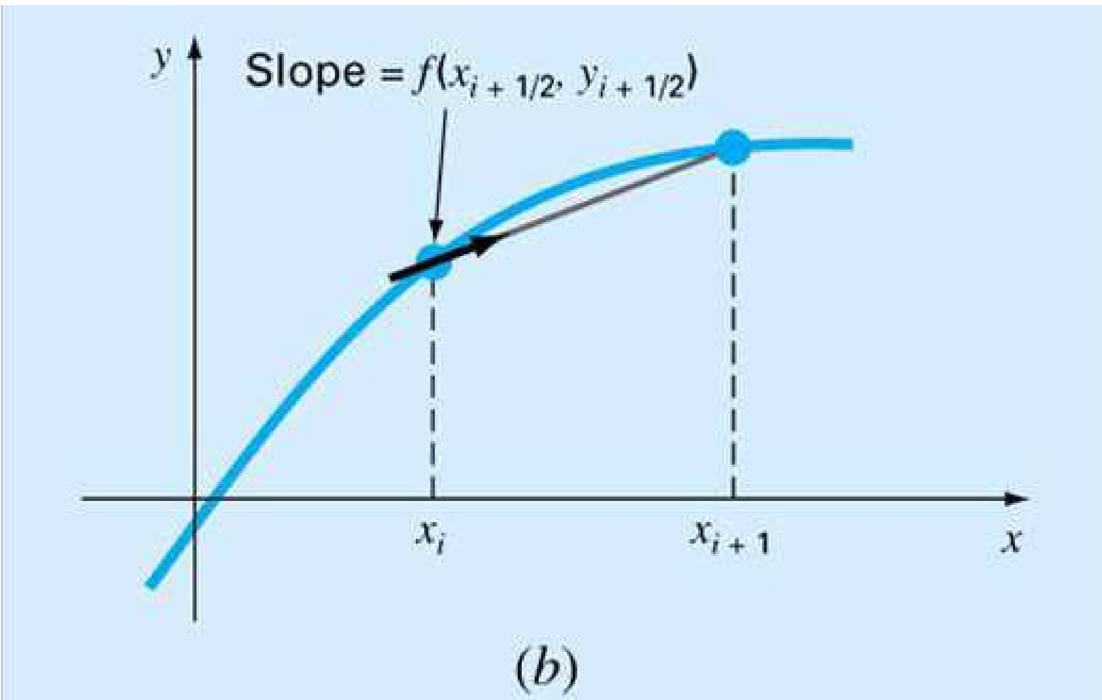

Figure 3: Illustration of the midpoint method

Fourth-order Runge-Kutta methods

Fourth-order RK methods have the form

$$y_{i+1} = y_i + (a_1 k_1 + a_2 k_2 + a_3 k_3 + a_4 k_4)h$$

Similar to the second-order RK methods, there are an infinite number of versions of fourth-order RK methods. The most commonly used form is

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h$$

where

$$\begin{aligned} k_1 &= f(x_i, y_i) \\ k_2 &= f(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h) \\ k_3 &= f(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_2 h) \\ k_4 &= f(x_i + h, y_i + k_3 h) \end{aligned}$$

Example: Use the classical fourth-order RK method to integrate

$$f(x, y) = -2x^3 + 12x^2 - 20x + 8.5$$

using a step size of $h = 0.5$ and an initial condition of $y = 1$ at $x = 0$.

Solution: $i = 0$, $x_0 = 0$, $y_0 = 1$.

$k_1 = f(x_0, y_0) = f(0, 1) = 8.5$

$k_2 = f(x_0 + \frac{h}{2}, y_0 + \frac{1}{2}k_1 h) = f(0.25, 3.125) = 4.21875$

$k_3 = f(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2 h) = f(0.25, 1 + \frac{1}{2} \times 4.21875 \times 0.5) = f(0.25, 2.0547) = 4.21875$

$k_4 = f(x_0 + h, y_0 + k_3 h) = f(0.25, 1 + 4.21875 \times 0.5) = 1.25$

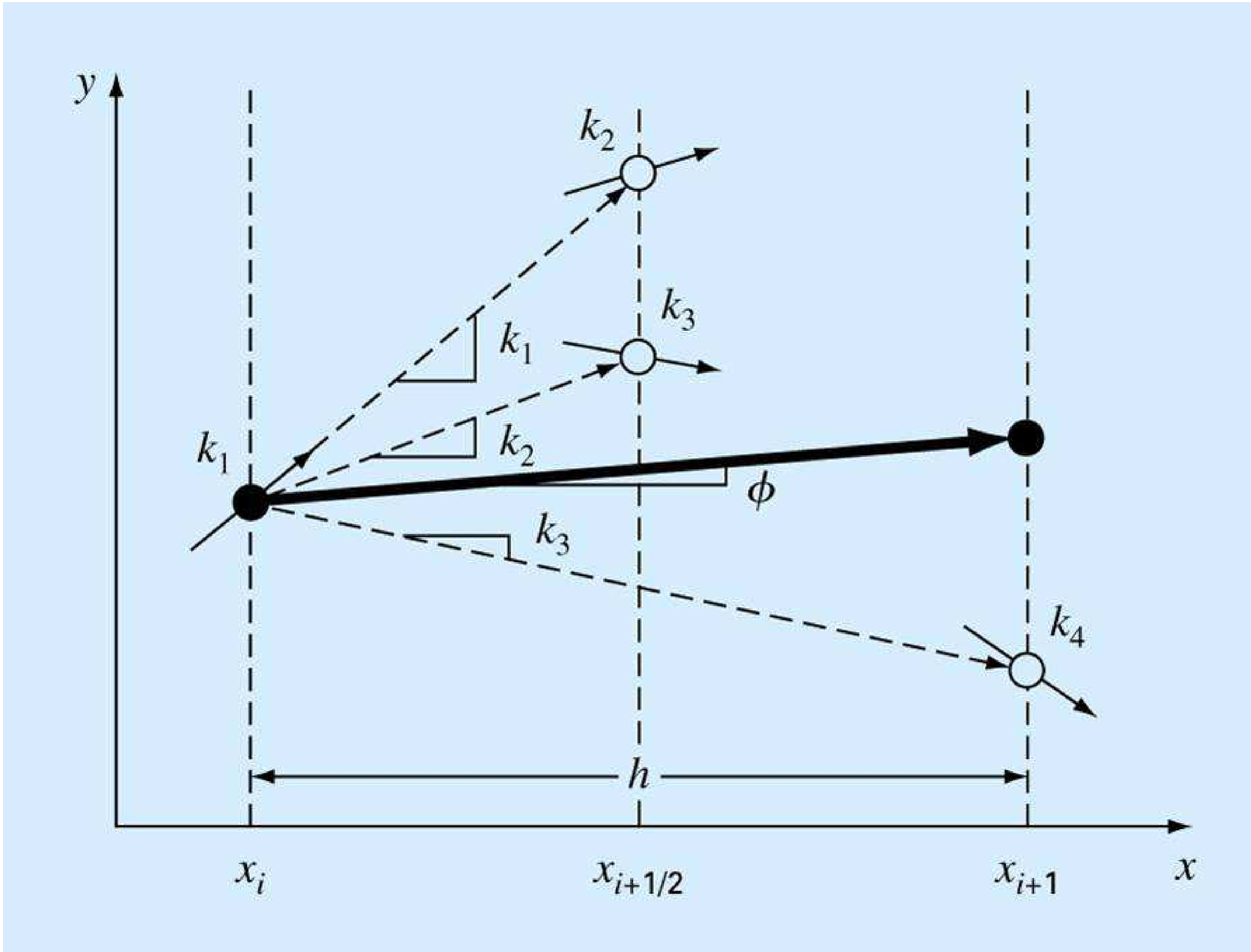

Figure 4: Illustration of slope estimates in the 4th order RK method

$x_1 = x_0 + h = 0.5$,
$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h = 1 + \frac{1}{6}(8.5 + 2 \times 4.21875 + 3 \times 4.21875 + 1.25) \times 0.5 = 3.21875$

This is exactly same as the true value ($y(0.5) = 3.21875$, see the previous example).

Because $y(x)$ is a fourth-order polynomial, the fourth-order RK methods give exact solution.

Example: Given $\frac{dy}{dx} = 4e^{0.8x} - 0.5y$, and $y(0) = 2$, (1) find $y(0.5)$ using analytical method, and (2) find $y(0.5)$ using the classical 4-th order RK method with step size $h = 0.5$.

Solution:

Analytical method:

(1) $\frac{dy}{dx} + 0.5y = 4e^{0.8x}$
$y_p = Ae^{0.8x}$, $\frac{dy_p}{dx} = 0.8Ae^{0.8x}$, then
$A \times 0.8e^{0.8x} + 0.5Ae^{0.x} = 4e^{0.8x}$, $A = \frac{40}{13}$
$y_p = \frac{40}{13}e^{0.8x}$.
(2) $\frac{dy_h}{dx} + 0.5h_h = 0$, then $\frac{dy_h}{dx} = -0.5y$, $\frac{dy_h}{0.5y_h} = -dx$, or $\int \frac{dy_h}{0.5y_h} = -\int dx$. Then
$\frac{1}{0.5}\ln y_h = -x + C'$, $\ln y_h = -0.5x + C''$, and
$y_h = e^{-0.5x + C''} = Ce^{-0.5x}$.
(3) $y = h_h + y_p = Ce^{-0.5x} + \frac{40}{13}e^{0.8x}$, with $(x_0, y_0) = (0, 2)$, $2 = C + \frac{40}{13}$, $C = -\frac{14}{13}$, and

$y = -\frac{14}{13}e^{-0.5x} + \frac{40}{13}e^{0.8x}$
(4) When $x = 0.5$, $y = -\frac{14}{13}e^{-0.25} + \frac{40}{13}e^{0.4} = 3.7515$
Classical 4-th order RK method:
$\frac{dy}{dx} = 4e^{0.8x} - 0.5y$, $f(x, y) = 4e^{0.8x} - 0.5y$, then
$k_1 = f(x_0, y_0) = f(0, 2) = 4e^0 - 0.5 \times 2 = 3$
$k_2 = f(x_0 + \frac{h}{2}, y_0 + \frac{1}{2}k_1 h) = f(0.25, 2 + \frac{1}{2} \times 3 \times 0.5) = f(0.25, 2.75) = 4 \times e^{0.8\times 0.25} - 0.5 \times 2.75 = 3.5106$
$k_3 = f(x_0 + \frac{h}{2}, y_0 + \frac{1}{2}k_2 h) = f(0.25, 2 + \frac{1}{2} \times 3.5106 \times 0.5) = f(0.25, 2.8777) = 3.4468$
$k_4 = f(x_0 + h, y_0 + k_3 h) = f(0.5, 2 + \times 3.4468 \times 0.5) = f(0.5, 3.7234) = 4.1056$
$x_1 = x_0 + h = 0.5$, $y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h = 2 + \frac{1}{6}(3 + 2 \times 3.5106 + 2 \times 3.4468 + 4.1056) \times 0.5 = 3.75167$
$\epsilon_t = 3.97 \times 10^{-5}$.

<u>n-th order RK methods</u>

- Accurate to n-th order polynomial
- Equivalent t n-th order Taylor series
- Does not require to evaluate derivatives

5 Systems of ODEs

For a system of simultaneous ODEs like

$$\begin{aligned}
\frac{dy_1}{dx} &= f_1(x, y_1, y_2, \cdots, y_n) \\
\frac{dy_2}{dx} &= f_2(x, y_1, y_2, \cdots, y_n) \\
&\vdots \\
\frac{dy_n}{dx} &= f_n(x, y_1, y_2, \cdots, y_n)
\end{aligned}$$

The solution of such a system requires that n initial conditions be known at the starting value of x, i.e., when $x = x_0$, the corresponding values of y_i, for all $i = 1, 2, \cdots, n$ are all known.

All the numerical methods we have discussed for single equations can be extended to solve a system of ODEs.

Example: Using Euler's method to solve the following set of ODEs:

$$\begin{aligned}
\frac{dy_1}{dx} &= -0.5y_1 \\
\frac{dy_2}{dx} &= 4 - 0.3y_2 - 0.1y_1
\end{aligned}$$

assuming that $x = 0$, $y_1 = 4$, and $y_2 = 6$. Integrate to $x = 2$ with a step size of 0.5.

Solution:

$$y_{i+1,1} = y_{i,1} + f_1(x_i, y_{i,1}, y_{i,2})$$
$$y_{i+1,2} = y_{i,2} + f_2(x_i, y_{i,2}, y_{i,2})$$

where $f_1(x, y_1, y_2) = -0.5y_1$, and $f_2(x, y_1, y_2) = 4 - 0.3y_2 - 0.1y_1$.

When $i = 0$, $x_1 = x_0 + h = 0.5$,

$y_{1,1} = y_{0,1} + f_1(x_0, y_{0,1}, y_{0,2})h = 4 + f_1(0, 4, 6) = 4 - 0.5 \times 4 \times 0.5 = 3$

$y_{1,2} = y_{0,2} + f_2(x_0, y_{0,1}, y_{0,2})h = 6 + f_2(0, 4, 6) = 6 + (4 - 0.3 \times 6 - 0.1 \times 4) \times 0.5 = 6.9$

When $i = 1$, $x_2 = x_1 + h = 1$

$y_{2,1} = y_{1,1} + f_1(x_1, y_{1,1}, y_{1,2})h = 3 + f_1(0.5, 3, 6.9) = 2.25$

$y_{2,2} = y_{1,2} + f_2(x_1, y_{1,1}, y_{1,2})h = 6.9 + f_2(0.5, 3, 6.9) = 7.715$

Example: Using the classical fourth-order RK method to solve the ODEs from the previous example.

Solution:

$$y_{i+1,1} = y_{i,1} + \frac{1}{6}(k_{1,1} + 2k_{1,2} + 2k_{1,3} + k_{1,4})h$$
$$y_{i+1,2} = y_{i,2} + \frac{1}{6}(k_{2,1} + 2k_{2,2} + 2k_{2,3} + k_{2,4})h$$

where

$$\begin{aligned}
k_{1,1} &= f_1(x_i, y_{i,1}, y_{i,2}) \\
k_{2,1} &= f_1(x_i + \frac{1}{2}h, y_{i,1} + \frac{1}{2}k_{1,1}h, y_{i,2} + \frac{1}{2}k_{1,2}h) \\
k_{3,1} &= f_1(x_i + \frac{1}{2}h, y_{i,1} + \frac{1}{2}k_{2,1}h, y_{i,2} + \frac{1}{2}k_{2,2}h) \\
k_{4,1} &= f_1(x_i + h, y_{i,1} + k_{3,1}h, y_{i,2} + k_{3,2}h)
\end{aligned}$$

and

$$\begin{aligned}
k_{1,2} &= f_2(x_i, y_{i,1}, y_{i,2}) \\
k_{2,2} &= f_2(x_i + \frac{1}{2}h, y_{i,1} + \frac{1}{2}k_{1,1}h, y_{i,2} + \frac{1}{2}k_{1,2}h) \\
k_{3,2} &= f_2(x_i + \frac{1}{2}h, y_{i,1} + \frac{1}{2}k_{2,1}h, y_{i,2} + \frac{1}{2}k_{2,2}h) \\
k_{4,2} &= f_2(x_i + h, y_{i,1} + k_{3,1}h, y_{i,2} + k_{3,2}h)
\end{aligned}$$

6 Multistep Methods

All previous methods are one-step methods which utilize information at a single point x_i to predict a value of the dependent variable y_{i+1} at a future point x_{i+1}. The multistep methods are based on the insight that, once the computation has begun, information from previous points can be used to estimate the function values at a future point.

The non-self-starting Heun method

This method a *predictor* and a *corrector* as

$$\text{Predictor: } y_{i+1}^0 = y_{i-1}^m + f(x_i, y_i^m)2h$$

$$\text{Corrector: } y_{i+1}^j = y_i^m + \frac{f(x_i, y_i^m) + f(x_{i+1}, y_{i+1}^{j-1})}{2}h$$

$$(\text{for } \; j = 1, 2, \ldots, m)$$

where the corrector is applied iteratively from $j = 1$ to m to obtain refined solutions. The approximate percentage relative error is

$$|\epsilon_a| = |\frac{y_{i+1}^j - y_{i+1}^{j-1}}{y_{i+1}^j}| \times 100\%$$

The iterations are terminated if ϵ_a is less than a prespecified error tolerance ϵ_s. The method is not self-starting because it involves a previous value of the dependent variable y_{i-1}.

Example: Use the non-self-starting Heun method to integrate

$$y^{'} = 4e^{0.8x} - 0.5y$$

using a step size of $h = 1.0$ and an initial condition of $y = 2$ at $x = 0$. Additional information is required for the multistep method: $y = -0.3929953$ at $x = -1$.

Solution: $x_{-1} = -1$, $y_{-1} = -0.3929953$; $x_0 = 0$, $y_0 = 2$.

Step 1: $x_1 = x_0 + h = 1$.
The predictor is used to extrapolate linearly from x_{-1} to x_1:

$$y_1^0 = y_{-1} + f(x_0, y_0)2h = -0.3929953 + (4e^{0.8\times 0} - 0.5 \times 2) \times 2 \times 1 = 5.607005$$

The corrector is then used to compute the value. When $j = 1$,

$$\begin{aligned} y_1^1 &= y_0 + \frac{f(x_0, y_0) + f(x_1, y_1^0)}{2} h \\ &= 1 + \frac{4e^{0.8\times 0} - 0.5 \times 2 + 4e^{0.8\times 0} - 0.5 \times 5.607005}{2} = 6.549331 \end{aligned}$$

The approximate percentage relative error is

$$\epsilon_a = |\frac{y_1^1 - y_1^0}{y_1^1}| \times 100\% = 14.39\%$$

When $j = 2$,

$$\begin{aligned} y_2^1 &= y_0 + \frac{f(x_0, y_0) + f(x_1, y_1^1)}{2} h \\ &= 1 + \frac{4e^{0.8\times 0} - 0.5 \times 2 + 4e^{0.8\times 0} - 0.5 \times 6.549331}{2} = 6.313749 \end{aligned}$$

The approximate percentage relative error is

$$\epsilon_a = |\frac{y_2^1 - y_1^1}{y_1^2}| \times 100\% = 3.73\%$$

The iteration can be repeated until ϵ_a is below a prespecified value of ϵ_s. The iterations converge on a value of 6.360865.

Step 2: $x_2 = x_1 + h = 2$.
The predictor is:

$$y_2^0 = y_0 + f(x_1, y_1)2h = 2 + (4e^{0.8\times 1} - 0.5 \times 6.360865) \times 2 \times 1 = 13.44346$$

The correctors can be calculated similarly as in Step 1.